Completely and Totally Distributive Categories

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Totally Distributive Categories

- categories denoted $\mathscr{K}$ etc, assumed to be *locally small*, ie homs in **set**, more generally in $(\mathscr{V}, \otimes, I, \cdots)$

- $\mathscr{K}$ is *total* [S&W] if $\exists X \dashv Y : \mathscr{K} \to \widehat{\mathscr{K}} = \mathbf{Cat}(\mathscr{K}^{\mathrm{op}}, \mathbf{set})$

- total $\mathscr{K}$ is *totally distributive* TD [R&W] if $\exists W \dashv X : \widehat{\mathscr{K}} \to \mathscr{K}$

- if $\mathscr{C}$ small then $Y : \mathscr{C} \to \widehat{\mathscr{C}}$ has both left and right kan extensions. by [S&W] the right of these is $Y : \widehat{\mathscr{C}} \to \widehat{\widehat{\mathscr{C}}}$ so

THEOREM for small $\mathscr{C}$, $\widehat{\mathscr{C}}$ is TD.

LEMMA if $\mathscr{K}$ is TD and

$$\mathscr{L} \begin{array}{c} \xrightarrow{\;j\;} \\ \bot \\ \xleftarrow{\;a\;} \\ \bot \\ \xrightarrow[\;i\;]{} \end{array} \mathscr{K} \quad (\text{or} \quad \mathscr{L} \begin{array}{c} \xleftarrow{\;a\;} \\ \bot \\ \xrightarrow{\;i\;} \\ \bot \\ \xleftarrow[\;b\;]{} \end{array} \mathscr{K} \quad \text{with } a_! \dashv \widehat{a}\ldots)$$

then $\mathscr{L}$ is TD.

Proof. in either case, $a \dashv i$ gives $X_{\mathscr{L}} \cong aX_M\widehat{a}$.
if $j \dashv a$ then $W_{\mathscr{L}} \cong \widehat{j}W_{\mathscr{K}}j$.
if $i \dashv b$ then $W_{\mathscr{L}} \cong a_!W_{\mathscr{K}}i$ if $a_! \dashv \widehat{a}\ldots$

COROLLARY if $\mathscr{L}$ is a lex CCD lattice (=: stably supercontinuous frame = regular projective in **frm** [B&N]) then $\mathbf{shv}(\mathscr{L})$ is TD.

Proof. $\Downarrow \dashv \bigvee : D\mathscr{L} \to \mathscr{L}$ in **frm** ($\Downarrow \dashv \bigvee \dashv \downarrow$ in **ord**) provides

$$\mathbf{shv}(\mathscr{L}) \; \begin{array}{c} \rightarrowtail \\ \xleftarrow{\;\mathbf{shv}(\bigvee)\;} \\ \rightarrowtail \\ {\scriptstyle \mathbf{shv}(\downarrow)} \end{array} \; \mathbf{shv}(D\mathscr{L}) \cong \widehat{\mathscr{L}}$$

THEOREM if $\mathscr{K}$ is TD then $\mathscr{K}$ is cototal, so that by [RJW]

$$\begin{array}{ccc} \widehat{\mathscr{K}} & \underset{(-)^{+}}{\overset{(-)^{-}}{\rightleftarrows}}\ \scriptstyle{\top} & (\widehat{\mathscr{K}^{\mathrm{op}}})^{\mathrm{op}} \\ & {\scriptstyle X}\searrow \quad \swarrow & \\ & \mathscr{K} & \end{array} \qquad A \vdash Z = \text{Yoneda} : \mathscr{K} \to (\widehat{\mathscr{K}^{\mathrm{op}}})^{\mathrm{op}}$$

Proof. $W \dashv X$ exhibits $\mathscr{K}$ as a full coreflective subcategory of a (large) power of **set**.

Completely Distributive Categories

- $\mathscr{P}\mathscr{K}$ = full subcat of $\widehat{\mathscr{K}}$ det by small colimits of representables.
- $\mathscr{P}\mathscr{K}$ is locally small and $\mathscr{P}\mathscr{K} = \widehat{\mathscr{K}}$ iff $\mathscr{K}$ is small.
- $\mathscr{P}\mathscr{K}$ is the free small colimit completion of $\mathscr{K}$.
- $\mathscr{P}\mathscr{K}$ need not be complete.
- $\mathscr{P}$:**Cat**$\to$**Cat** underlies a KZ (pseudo) monad.
- $\mathscr{K}$ is *(small) cocomplete* iff $Y{:}\mathscr{K}\to\mathscr{P}\mathscr{K}$ has a left adjoint.
- $\mathscr{R} = (\mathscr{P}(-)^{\mathrm{op}}))^{\mathrm{op}}$:**Cat**$\to$**Cat** underlies the free small limit completion (coKZ) monad.

• any limit in $\mathscr{P}\mathscr{K}$ is pointwise so $\mathscr{P}\mathscr{K} \twoheadrightarrow \widehat{\mathscr{K}}$ preserves any limits that exist in $\mathscr{P}\mathscr{K}$.

• $\mathscr{P}\mathscr{K}$ is complete if $\mathscr{K}$ is so. [D&L] [PJF]

Proof Sketch. I) $\mathscr{K}^{\mathrm{op}}$ locally presentable II) $\mathscr{R}\mathscr{C}$ small $\mathscr{C}$
III) $\mathscr{R}\mathscr{K}$ IV) $\mathscr{K}$ complete.

• for $\mathscr{K}$ and $\mathscr{L}$ complete, $F{:}\mathscr{K} \to \mathscr{L}$ is continuous if and only if $\mathscr{P}F{:}\mathscr{P}\mathscr{K} \to \mathscr{P}\mathscr{L}$ is so. [D&L]

• $Y_{\mathscr{K}}{:}\mathscr{K} \to \mathscr{P}\mathscr{K}$ is continuous for complete $\mathscr{K}$,
$\mathscr{P}Y_{\mathscr{K}} \dashv M_{\mathscr{K}} \dashv Y_{\mathscr{P}\mathscr{K}}$ so $M_{\mathscr{K}}$ is continuous for complete $\mathscr{K}$.

• $(\mathscr{P}, Y, M)$ lifts from **Cat** to $\mathbf{Cat}^{\mathscr{R}}$. [D&L]

COROLLARY there is a distributive law $\rho{:}\mathscr{R}\mathscr{P} \to \mathscr{P}\mathscr{R}$.

• $\mathscr{K}$ is *completely distributive* CD if $\mathscr{K}$ is a ρ-algebra, ie if $\mathscr{K}$ is a $\mathscr{P}$-algebra and $\mathscr{K}$ is an $\mathscr{R}$-algebra and $\mathscr{P}\mathscr{K} \to \mathscr{K}$ is an $\mathscr{R}$-homomorphism. ie if $\mathscr{K}$ is cocomplete and complete and assignment of colimits preserves all (small) limits.

REMARK a stronger definition would require that $\mathscr{P}\mathscr{K} \to \mathscr{K}$ have a left adjoint, equivalently that $(\mathscr{K}, \mathscr{P}\mathscr{K} \to \mathscr{K})$ have a $\mathscr{P}$-coalgebra structure for $\mathscr{P}$ seen as a comonad on $\mathbf{Cat}^{\mathscr{P}}$.

REMARK a ρ-algebra $\mathscr{K}$ has *limits distributing* over *colimits* in the terminology of Beck. There is also a distributive law $\lambda{:}\mathscr{P}\mathscr{R} \to \mathscr{R}\mathscr{P}$ and a λ-algebra has *colimits distributing over colimits*. $\mathscr{K}$ is a λ-algebra if and only if $\mathscr{K}^{\mathrm{op}}$ is a ρ-algebra.

THEOREM TD implies CD.

Proof. $\mathscr{P}$- [$\mathscr{R}$-] structure restriction of X [A]; $\mathscr{P}\mathscr{K} \twoheadrightarrow \widehat{\mathscr{K}}$ cts.

Taxons

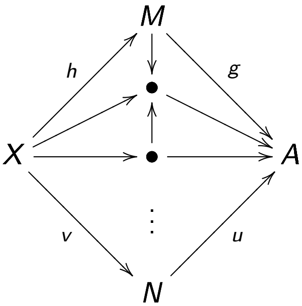
• recall that a category $\mathscr{K}$ is a (possibly large) set $|\mathscr{K}|$ together with a monad $\mathscr{K}:|\mathscr{K}|\to|\mathscr{K}|$ in **Mat**, bicategory of **set**- [$\mathscr{V}$-] valued matrices.

• a *taxon* [KOS] $\mathbf{T}$ is a set $|\mathbf{T}|$ together with an *interpolad* $\mathbf{T}{:}|\mathbf{T}|\to|\mathbf{T}|$ in **Mat**. ie $\mathbf{T} = (\mathbf{T}{:}|\mathbf{T}|\to|\mathbf{T}|, \mu{:}\mathbf{TT}\to\mathbf{T})$ where

$$\mathbf{TTT} \underset{\mu\mathbf{T}}{\overset{\mathbf{T}\mu}{\rightrightarrows}} \mathbf{TT} \xrightarrow{\ \mu\ } \mathbf{T}$$

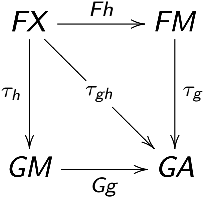
is a coequalizer. (a taxon 'is' a category with identity requirement replaced by interpolativity of composition μ.) for all $f{:}X\to A$ in $\mathbf{T}$, there exists a unique $g\otimes h$ such that $f = gh$:

$g \otimes h$



- a *functor* $F\colon \mathbf{S} \rightarrowtail \mathbf{T}$ …
- a *natural transformation* $\tau\colon F \rightarrowtail G\colon \mathbf{S} \rightarrowtail \mathbf{T}$

$f : X \rightarrowtail A \mapsto \tau_f\colon FX \rightarrowtail GA$ such that, for all $X \xrightarrow{h} M \xrightarrow{g} A$,



- with evident compositions these data form a 2-category $\mathbf{Tax}$.
- An *i-module* $\mathbf{M}{:}\mathbf{S}\to\mathbf{T}$ is a matrix $\mathbf{M}{:}|\mathbf{S}|\to|\mathbf{T}|$ together with mutually associative actions $\lambda{:}\mathbf{TM}\to\mathbf{M}$ and $\rho{:}\mathbf{MS}\to\mathbf{M}$ for which

$$\mathbf{TTM} \underset{\mathbf{T}\lambda}{\overset{\mu\mathbf{M}}{\rightrightarrows}} \mathbf{TM} \xrightarrow{\lambda} \mathbf{M} \qquad \& \qquad \mathbf{MSS} \underset{\rho\mathbf{S}}{\overset{\mathbf{M}\mu}{\rightrightarrows}} \mathbf{MS} \xrightarrow{\rho} \mathbf{M}$$

are coequalizers.

- small taxons, i-modules, and equivariant 2-cells form bicat $\mathbf{imod}$.
- proarrow equipment $(-)_*{:}\mathbf{tax}\to\mathbf{imod}$, $F_* = F_\bullet\mathbf{S}$;

$F_\bullet(T,S) = \mathbf{T}(T,FS)$; $\mathbf{imod}$ has all right liftings and extensions.

$$\begin{array}{ccccc} & \mathbf{cat} & \xrightarrow{(-)_*} & \mathbf{prof} & \\ \text{not full} & i \downarrow \dashv \uparrow k & & \downarrow i & \text{fully faithful} \\ & \mathbf{tax} & \xrightarrow[(-)_*]{} & \mathbf{imod} & \end{array}$$

- $\mathbf{imod}(i\mathbf{1},\mathbf{T}) \xrightarrow{\sim} \mathbf{Tax}(\mathbf{T}^{\mathrm{op}}, i\mathbf{set})$ (*objects* carry *mere* actions $i\mathbf{1}\to\mathbf{T}$)

THEOREM for $\mathbf{T}$ a small taxon, $\mathbf{Tax}(\mathbf{T}^{\mathrm{op}}, i\mathbf{set})$ is TD.

Proof Sketch. $\mathbf{T} \xrightarrow{Y_{\mathbf{T}}} i\mathbf{imod}(i\mathbf{1}, \mathbf{T})$ in $\mathbf{Tax}$ gives

$$\mathbf{T} \underset{Y_{\mathbf{T}}{}^{*}}{\overset{Y_{\mathbf{T}*}}{\rightleftarrows}} i\mathbf{imod}(i\mathbf{1}, \mathbf{T}) \qquad (Y_{\mathbf{T}}{}^{*} \dashv Y_{\mathbf{T}*})$$

in i-modules and hence

$$\begin{array}{ccc}
\mathbf{imod}(i\mathbf{1}, \mathbf{T}) & \begin{array}{c} \xrightarrow{\mathbf{imod}(i\mathbf{1}, Y_{\mathbf{T}*})} \\ \xleftarrow{\mathbf{imod}(i\mathbf{1}, Y_{\mathbf{T}}{}^{*})} \;\perp \\ \perp\; \xrightarrow[Y_{\mathbf{T}}{}^{*} \Rightarrow (-)]{} \end{array} & \mathbf{imod}(i\mathbf{1}, i\mathbf{imod}(i\mathbf{1}, \mathbf{T})) \\
& \searrow^{Y_{\mathbf{imod}(i\mathbf{1}, \mathbf{T})}} & \uparrow{\scriptstyle i_{\mathbf{1}, \mathbf{imod}(i\mathbf{1}, \mathbf{T})}} \\
& & \widehat{\mathbf{imod}(i\mathbf{1}, \mathbf{T})} = \mathbf{prof}(\mathbf{1}, \mathbf{imod}(i\mathbf{1}, \mathbf{T}))
\end{array}$$

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