Two 2-traces

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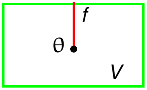
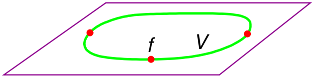
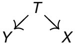
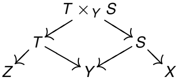
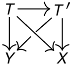
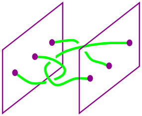
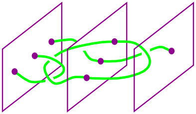
CT2010: Genova

$\mathrm{Tr}^{\searrow}(f) :=$ {


}

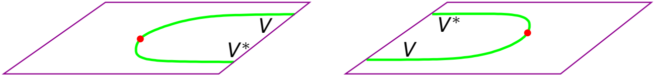
$\mathrm{Tr}^{\circlearrowright}(f) :=$


Examples of monoidal bicategories

	objects	1-morphisms	composition	2-morphisms	tensor
Span	Sets				$\times$
Bim	Algebras over $\mathbb{C}$	${}_B M_A$	${}_C N_B \otimes_B {}_B M_A$	$\mathrm{Hom}_{B,A}({}_B M_A, {}_B M'_A)$	$\otimes_{\mathbb{C}}$
$\mathcal{V}$-Mod	$\mathcal{V}$-cats	$\mathcal{C}^{\mathrm{op}} \otimes \mathcal{D} \to \mathcal{V}$	$\otimes_D$	$\mathcal{V}$-nat trans	$\otimes$
2-Tang	Pts in $\mathbb{R}^2$			cobordisms	$\coprod$
Var	Complex manifolds	$\mathcal{E}^\bullet \downarrow Y \times X$	convolution	$\mathrm{Ext}^\bullet_{Y\times X}(\mathcal{E}^\bullet, \mathcal{F}^\bullet)$	$\times$
DBim	Diff algs over $\mathbb{C}$	$\to {}_B M^i_A \to {}_B M^{i-1}_A \to$	$\otimes^{\mathrm{L}}_B$	$\mathrm{Ext}^\bullet_{B\times A^{\mathrm{op}}}({}_B M^\bullet_A, {}_B N^\bullet_A)$	$\otimes_{\mathbb{C}}$

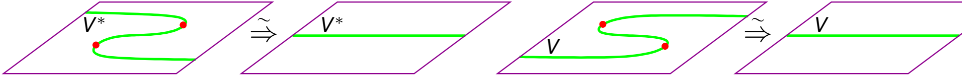
Ambiduals in a monoidal bicategory

In $\mathcal{C}$, an object V^* is left-dual to V if there exist 1-morphisms

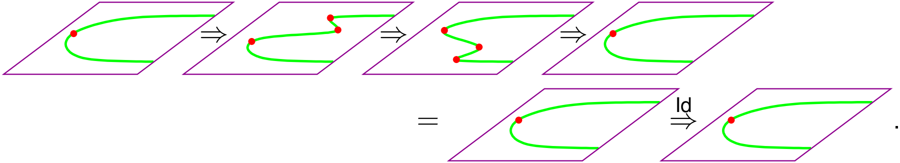


$\mathbf{1} \xleftarrow{\text{ev}} V^* \otimes V$ $\qquad$ $V \otimes V^* \xleftarrow{\text{coev}} \mathbf{1}$

and 2-isomorphisms

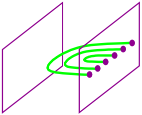
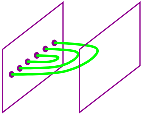


such that the Swallowtail Relations hold, e.g.,



If V is also left dual to V^* then V and V^* are ambidual.

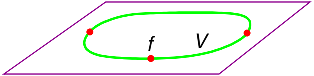
Examples of ambiduals in monoidal bicategories

	object	dual	evaluation	coevaluation
Span	X	X	$\star \swarrow X \xrightarrow{\Delta} X \times X$	$X \times X \xleftarrow{\Delta} X \searrow \star$
Bim	A	A^{op}	${}_{\mathbb{C}}A_{A^{\mathrm{op}} \otimes A}$	${}_{A \otimes A^{\mathrm{op}}}A_{\mathbb{C}}$
$\mathcal{V}$-Mod	$\mathcal{C}$	$\mathcal{C}^{\mathrm{op}}$	$\mathcal{C}^{\mathrm{op}} \otimes \mathcal{C} \otimes \star \xrightarrow{\mathrm{Hom}} \mathcal{V}$	$\star \otimes \mathcal{C} \otimes \mathcal{C}^{\mathrm{op}} \xrightarrow{\mathrm{Hom}} \mathcal{V}$
2-Tang				
Var	X	X	$\begin{array}{c}\mathcal{O}_\Delta \\ \downarrow \\ \star \times X \times X\end{array}$	$\begin{array}{c}\mathcal{O}_\Delta \\ \downarrow \\ X \times X \times \star\end{array}$
DBim	$A^\bullet$	$A^{\bullet\,\mathrm{op}}$	${}_{\mathbb{C}}A^\bullet_{A^{\bullet\,\mathrm{op}} \otimes A^\bullet}$	${}_{A^\bullet \otimes A^{\bullet\,\mathrm{op}}}A^\bullet_{\mathbb{C}}$

The round trace

If V has an ambidual and $V \xleftarrow{f} V$ define the round trace:

$\mathrm{Tr}^{\circlearrowright}(f) :=$



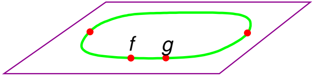
$\in$ 1-Hom$(\mathbf{1}, \mathbf{1})$.

Theorem (Trace property)

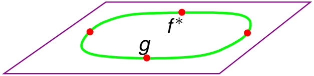
If $V \xleftarrow{f} W$ and $W \xleftarrow{g} V$ with f having a transpose $W^* \xleftarrow{f^*} V^*$ then

$$\mathrm{Tr}^{\circlearrowright}(f \circ g) \cong \mathrm{Tr}^{\circlearrowright}(g \circ f).$$

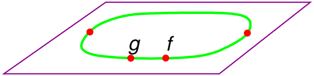
$\mathrm{Tr}^{\circlearrowright}(f \circ g) =$



$\overset{\sim}{\Longrightarrow}$



$\overset{\sim}{\Longrightarrow}$



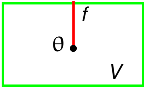
$= \mathrm{Tr}^{\circlearrowright}(g \circ f)$

The diagonal trace

This can be defined in a bicategory without monoidal structure.

If V is an object of a bicategory and $V \xleftarrow{f} V$ define the diagonal trace:

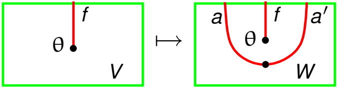
$$\mathrm{Tr}^{\searrow}(f) := \text{2-Hom}(\mathrm{Id}_V, f) = \left\{ \quad \right\}$$



Theorem (Trace property)

Given $W \xleftarrow{a} V$, $V \xleftarrow{a'} W$ and $a \circ a' \xLeftarrow{\eta} \mathrm{Id}_W$ there is a morphism:

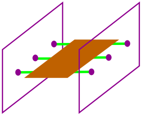
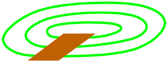
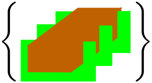
$$\mathrm{Tr}^{\searrow}(f) \xrightarrow{\eta_*} \mathrm{Tr}^{\searrow}(a \circ f \circ a')$$



If $W \xleftarrow{a} V$ is an adjoint equivalence then

$$\mathrm{Tr}^{\searrow}(f) \cong \mathrm{Tr}^{\searrow}(a \circ f \circ a^{-1}).$$

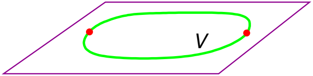
Examples of traces in monoidal bicategories

	object	endo, f	$\mathrm{Tr}^{\circlearrowleft}(f)$	$\mathrm{Tr}^{\searrow}(f)$
Span	X	$X \swarrow T \searrow X$	"loops in T"	"choice of loop at each $x \in X$"
Bim	A	${}_A M_A$	$M/\{ma - am\}$ coinvariants	$\{m \in M \mid am = ma\}$ invariants
$\mathcal{V}$-Mod	$\mathcal{C}$	$\mathcal{C}^{\mathrm{op}} \otimes \mathcal{C} \xrightarrow{F} \mathcal{V}$	$\int^{c} F(c, c)$	$\int_{c} F(c, c)$
2-Tang				
Var	X	$\mathcal{E}^\bullet \downarrow X \times X$	$\mathrm{HH}_\bullet(X, \mathcal{E}^\bullet)$	$\mathrm{HH}^\bullet(X, \mathcal{E}^\bullet)$
DBim	$A^\bullet$	${}_{A^\bullet} M^\bullet_{A^\bullet}$	$\mathrm{HH}_\bullet(A^\bullet, M^\bullet)$	$\mathrm{HH}^\bullet(A^\bullet, M^\bullet)$

Dimension

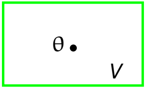
The dimension of an object can be defined to be the trace of the identity.

$$\mathrm{Dim}^{\circlearrowright}(V) := \mathrm{Tr}^{\circlearrowright}(\mathrm{Id}_V) =$$



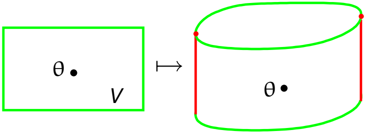
$$\in 1\text{-Hom}(\mathbf{1}, \mathbf{1})$$

$$\mathrm{Dim}^{\searrow}(V) := \mathrm{Tr}^{\searrow}(\mathrm{Id}_V) = 2\text{-Hom}(\mathrm{Id}_V, \mathrm{Id}_V) = \left\{ \quad \right\}$$

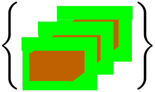


- $\mathrm{Dim}^{\searrow}(V)$ is a commutative monoid
- $\mathrm{Dim}^{\searrow}(V)$ acts on $\mathrm{Dim}^{\circlearrowright}(V)$

$$\mathrm{Dim}^{\searrow}(V) \to 2\text{-Hom}\left(\mathrm{Dim}^{\circlearrowright}(V), \mathrm{Dim}^{\circlearrowright}(V)\right)$$



Examples of dimensions in monoidal bicategories

	object, V	$\mathrm{Dim}^{\circlearrowright}(V)$	$\mathrm{Dim}^{\searrow}(V)$
Span	X	X	$\{\star\}$
Bim	A	$A/[A, A]$	Z(A)
$\mathcal{V}$-Mod	$\mathcal{C}$	$\int^{c} \mathcal{C}(c, c)$	$\mathcal{V}$-NAT $(\mathrm{Id}_{\mathcal{C}}, \mathrm{Id}_{\mathcal{C}})$
2-Tang			
Var	X	$\mathrm{HH}_{\bullet}(X)$	$\mathrm{HH}^{\bullet}(X)$
DBim	$A^{\bullet}$	$\mathrm{HH}_{\bullet}(A^{\bullet})$	$\mathrm{HH}^{\bullet}(A^{\bullet})$