

# Two 2-traces


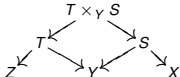

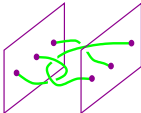
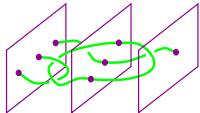
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CT2010: Genova

$$\mathrm{Tr}^{\searrow}(f) := \left\{ \begin{array}{c} \text{[Diagram: A green rectangle containing a black dot labeled } \theta \text{ and a red vertical line labeled } f \text{ extending upwards from } \theta \text{ to the top edge of the rectangle. The label } V \text{ is at the bottom right corner of the rectangle.]} \\ \end{array} \right\}$$

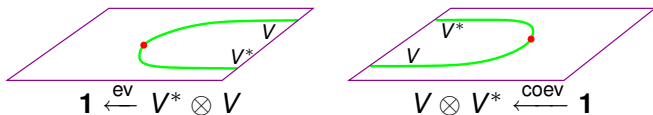
$$\mathrm{Tr}^{\circlearrowleft}(f) := \begin{array}{c} \text{[Diagram: A purple parallelogram containing a green oval with three red dots on its boundary. One red dot is at the bottom center and is labeled } f \text{, another is at the top right, and the third is at the top left. The label } V \text{ is at the bottom right of the oval.]} \\ \end{array}$$

# Examples of monoidal bicategories

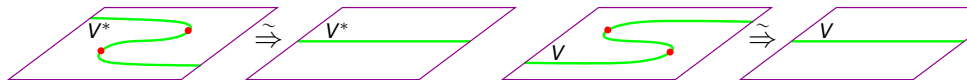
	objects	1-morphisms	composition	2-morphisms	tensor
Span	Sets				$\times$
Bim	Algebras over $\mathbb{C}$	${}_B M_A$	${}_C N_B \otimes_B {}_B M_A$	$\text{Hom}_{B,A}({}_B M_A, {}_B M'_A)$	$\otimes_{\mathbb{C}}$
$\mathcal{V}$ -Mod	$\mathcal{V}$ -cats	$\mathcal{C}^{\text{op}} \otimes \mathcal{D} \rightarrow \mathcal{V}$	$\otimes_D$	$\mathcal{V}$ -nat trans	$\otimes$
2-Tang	Pts in $\mathbb{R}^2$			cobordisms	$\sqcup$
Var	Complex manifolds	$\mathcal{E}^\bullet$ $\downarrow$ $Y \times X$	convolution	$\text{Ext}_{Y \times X}^\bullet(\mathcal{E}^\bullet, \mathcal{F}^\bullet)$	$\times$
DBim	Diff algs over $\mathbb{C}$	$\rightarrow {}_B M_A^i \rightarrow {}_B M_A^{i-1} \rightarrow$	$\otimes_B^L$	$\text{Ext}_{B \times A^{\text{op}}}^\bullet({}_B M_A^\bullet, {}_B N_A^\bullet)$	$\otimes_{\mathbb{C}}$

# Ambiduals in a monoidal bicategory

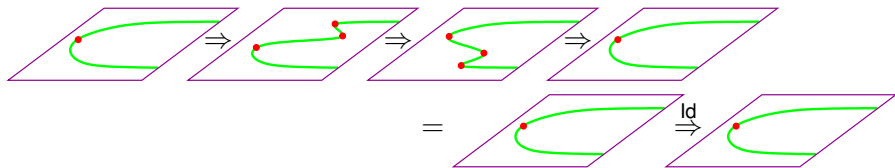
In  $\mathcal{C}$ , an object  $V^*$  is left-dual to  $V$  if there exist 1-morphisms



and 2-isomorphisms



such that the Swallowtail Relations hold, e.g.,



If  $V$  is also left dual to  $V^*$  then  $V$  and  $V^*$  are **ambidual**.

# Examples of ambiduals in monoidal bicategories

	object	dual	evaluation	coevaluation
Span	$X$	$X$		
Bim	$A$	$A^{\text{op}}$	$\mathbb{C}A_{A^{\text{op}} \otimes A}$	$A \otimes A^{\text{op}} A_{\mathbb{C}}$
$\mathcal{V}$ -Mod	$\mathcal{C}$	$\mathcal{C}^{\text{op}}$	$\mathcal{C}^{\text{op}} \otimes \mathcal{C} \otimes \star \xrightarrow{\text{Hom}} \mathcal{V}$	$\star \otimes \mathcal{C} \otimes \mathcal{C}^{\text{op}} \xrightarrow{\text{Hom}} \mathcal{V}$
2-Tang				
Var	$X$	$X$	$\begin{array}{c} \mathcal{O}_{\Delta} \\ \downarrow \\ \star \times X \times X \end{array}$	$\begin{array}{c} \mathcal{O}_{\Delta} \\ \downarrow \\ X \times X \times \star \end{array}$
DBim	$A^{\bullet}$	$A^{\bullet \text{op}}$	$\mathbb{C}A_{A^{\bullet \text{op}} \otimes A^{\bullet}}$	$A^{\bullet} \otimes A^{\bullet \text{op}} A^{\bullet}_{\mathbb{C}}$

# The round trace

If  $V$  has an ambidual and  $V \xleftarrow{f} V$  define the **round trace**:

$$\mathrm{Tr}^{\circlearrowleft}(f) := \text{[Diagram: A purple parallelogram containing a green oval with three red dots and the label 'f V']} \in \mathbf{1}\text{-Hom}(\mathbf{1}, \mathbf{1}).$$

## Theorem (Trace property)

If  $V \xleftarrow{f} W$  and  $W \xleftarrow{g} V$  with  $f$  having a transpose  $W^* \xleftarrow{f^*} V^*$  then

$$\mathrm{Tr}^{\circlearrowleft}(f \circ g) \cong \mathrm{Tr}^{\circlearrowleft}(g \circ f).$$

$$\begin{aligned} \mathrm{Tr}^{\circlearrowleft}(f \circ g) &= \text{[Diagram: Purple parallelogram with green oval, red dots, and labels 'f g']} \xrightarrow{\sim} \text{[Diagram: Purple parallelogram with green oval, red dots, and labels 'f*' and 'g']} \\ &\xrightarrow{\sim} \text{[Diagram: Purple parallelogram with green oval, red dots, and labels 'g f']} = \mathrm{Tr}^{\circlearrowleft}(g \circ f) \end{aligned}$$

## The diagonal trace

This can be defined in a bicategory **without** monoidal structure.

If  $V$  is an object of a bicategory and  $V \xleftarrow{f} V$  define the **diagonal trace**:

$$\mathrm{Tr}^{\searrow}(f) := 2\text{-Hom}(\mathrm{Id}_V, f) = \left\{ \begin{array}{|c|} \hline \text{\scriptsize } \theta \bullet \text{\scriptsize } \\ \hline \text{\scriptsize } V \\ \hline \end{array} \right\}$$

### Theorem (Trace property)


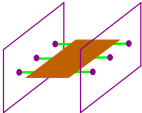
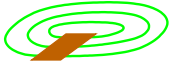
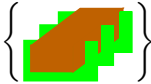
Given  $W \xleftarrow{a} V$ ,  $V \xleftarrow{a'} W$  and  $a \circ a' \xleftarrow{\eta} \mathrm{Id}_W$  there is a morphism:

$$\mathrm{Tr}^{\searrow}(f) \xrightarrow{\eta_*} \mathrm{Tr}^{\searrow}(a \circ f \circ a')$$

If  $W \xleftarrow{a} V$  is an adjoint equivalence then

$$\mathrm{Tr}^{\searrow}(f) \cong \mathrm{Tr}^{\searrow}(a \circ f \circ a^{-1}).$$

# Examples of traces in monoidal bicategories

	object	endo, $f$	$\text{Tr}^{\circlearrowleft}(f)$	$\text{Tr}^{\searrow}(f)$
Span	$X$	$X \xleftarrow{T} X$	“loops in $T$ ”	“choice of loop at each $x \in X$ ”
Bim	$A$	${}_A M_A$	$M/\{ma - am\}$ coinvariants	$\{m \in M \mid am = ma\}$ invariants
$\mathcal{V}$ -Mod	$\mathcal{C}$	$\mathcal{C}^{\text{op}} \otimes \mathcal{C} \xrightarrow{F} \mathcal{V}$	$\int^{\mathcal{C}} F(c, c)$	$\int_{\mathcal{C}} F(c, c)$
2-Tang				
Var	$X$	$\begin{array}{c} \mathcal{E}^{\bullet} \\ \downarrow \\ X \times X \end{array}$	$\text{HH}_{\bullet}(X, \mathcal{E}^{\bullet})$	$\text{HH}^{\bullet}(X, \mathcal{E}^{\bullet})$
DBim	$A^{\bullet}$	${}_{A^{\bullet}} M_{A^{\bullet}}$	$\text{HH}_{\bullet}(A^{\bullet}, M^{\bullet})$	$\text{HH}^{\bullet}(A^{\bullet}, M^{\bullet})$

# Dimension

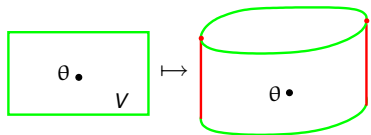
The dimension of an object can be defined to be the trace of the identity.

$$\text{Dim}^{\circlearrowleft}(V) := \text{Tr}^{\circlearrowleft}(\text{Id}_V) = \text{[Diagram: a purple parallelogram with a green oval inside labeled } V \text{ and two red dots on the oval]} \in \mathbf{1}\text{-Hom}(\mathbf{1}, \mathbf{1})$$

$$\text{Dim}^{\searrow}(V) := \text{Tr}^{\searrow}(\text{Id}_V) = \mathbf{2}\text{-Hom}(\text{Id}_V, \text{Id}_V) = \left\{ \begin{array}{c} \text{[Diagram: a green rectangle with a dot labeled } \theta \text{ and } V \text{ at the bottom]} \end{array} \right\}$$



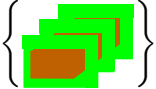
- ▶  $\text{Dim}^{\searrow}(V)$  is a commutative monoid
- ▶  $\text{Dim}^{\searrow}(V)$  acts on  $\text{Dim}^{\circlearrowleft}(V)$

$$\text{Dim}^{\searrow}(V) \rightarrow \mathbf{2}\text{-Hom}(\text{Dim}^{\circlearrowleft}(V), \text{Dim}^{\circlearrowleft}(V))$$





# Examples of dimensions in monoidal bicategories

	object, $V$	$\text{Dim}^{\circlearrowleft}(V)$	$\text{Dim}^{\searrow}(V)$
Span	$X$	$X$	$\{\star\}$
Bim	$A$	$A/[A, A]$	$Z(A)$
$\mathcal{V}$ -Mod	$\mathcal{C}$	$\int^{\mathcal{C}} \mathcal{C}(c, c)$	$\mathcal{V}\text{-NAT}(\text{Id}_{\mathcal{C}}, \text{Id}_{\mathcal{C}})$
2-Tang			
Var	$X$	$\text{HH}_{\bullet}(X)$	$\text{HH}^{\bullet}(X)$
DBim	$A^{\bullet}$	$\text{HH}_{\bullet}(A^{\bullet})$	$\text{HH}^{\bullet}(A^{\bullet})$