Approximation in quantale-enriched categories

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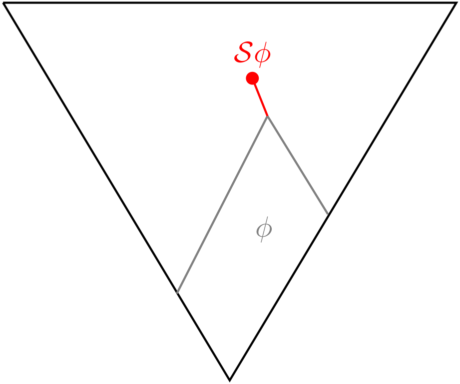
A motivation from domain theory

A dcpo is a poset where every ideal (directed, lower subset) has a supremum.



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But

$$\downarrow := \mathrm{y}$$

is just the Yoneda embedding. Hence a poset X is a dcpo iff the Yoneda embedding $\mathrm{y}\colon X \to \mathbf{Idl}X$ has a left adjoint $\mathcal{S}\colon \mathbf{Idl}X \to X$:

$$\mathcal{S} \dashv \mathrm{y}.$$

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A dcpo is continuous if $x = \mathcal{S}(\Downarrow x)$, for all $x \in X$.

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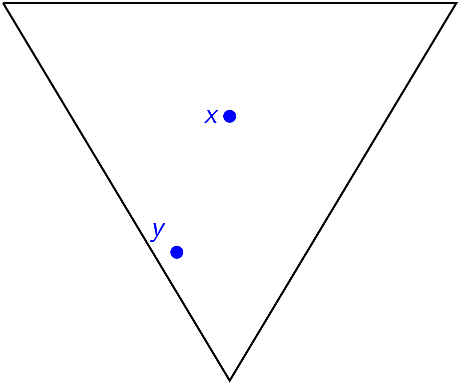

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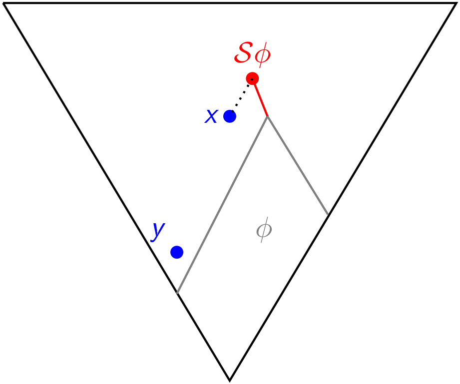

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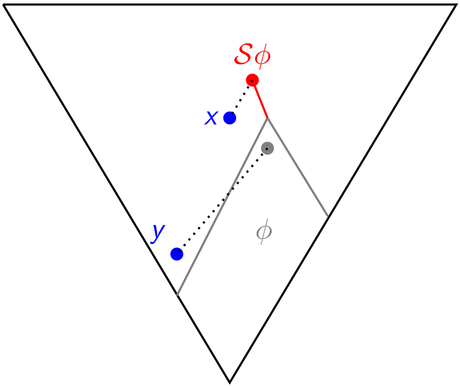

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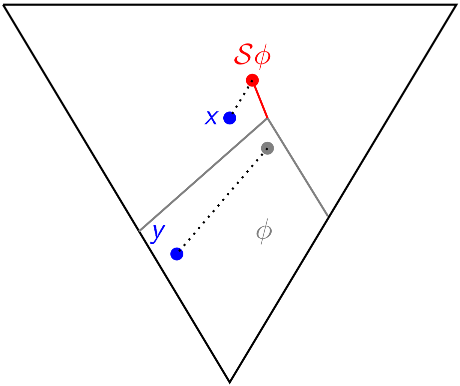

DEF.: $y \ll x$ iff $\forall \phi \, (x \leqslant \mathcal{S}\phi \iff y \in \phi)$.

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- In the lattice of opens of a locally compact Hausdorff space,

$$O \ll U \quad \text{iff} \quad O \subseteq K \subseteq U \text{ for some compact } K.$$

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Hence a dcpo X is continuous iff $\mathcal{S}\colon \mathbf{Idl}X \to X$ has a left adjoint $\Downarrow\colon X \to \mathbf{Idl}X$

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J-continuous $\mathcal{Q}$-categories

We can further generalize the situation by:

- replacing $\mathbf{2}$ by a commutative, unital quantale $\mathcal{Q}$,
- replacing the class **Idl** by some other class of weights $J \subseteq \widehat{X}$.

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DEFINITION

A $\mathcal{Q}$-category X is J-cocomplete and J-continuous if

$$\Downarrow \dashv \mathcal{S} \dashv \mathrm{y}$$

for

$$\Downarrow : X \to JX,$$

$$\mathcal{S} : JX \to X,$$

$$\mathrm{y} : X \to JX.$$

The simplest examples of J-cocomplete J-continuous $\mathcal{Q}$-categories

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- For $J = \mathbf{Idl}$, $\mathcal{Q} = \mathbf{2}$, we get continuous dcpos.
- For $J = \widehat{X}$, $\mathcal{Q} = \mathbf{2}$, we get completely distributive complete lattices.

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EXAMPLE: For $\mathcal{Q} = ([0, \infty], +)$, assuming X is symmetric, we recover complete metric spaces.

EXAMPLE: For $\mathcal{Q} = \mathbf{2}$, we recover continuous dcpos.

Three short stories about $\mathbb{A}$-cocomplete and $\mathbb{A}$-cocomplete $\mathbb{A}$-continuous $\mathcal{Q}$-categories

1. On embedding into continuous dcpos.
2. On a duality theorem.
3. On fixed points of endofunctors.

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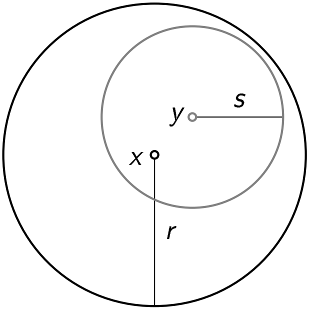

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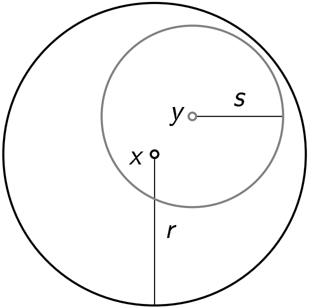
$\mathbf{B}X := \{\langle x, r\rangle \mid x \in X \text{ and } r \geqslant 0\} \subseteq X \times \mathbb{R}_+$
$\langle x, r\rangle \leqslant \langle y, s\rangle \;\text{ iff }\; X(x, y) + s \leqslant r$
$X \cong \{\langle x, 0\rangle \mid x \in X\}$ $(= \max(\mathbf{B}X)$ providing X is $T1)$.



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THEOREM If X is, in addition, T_1, then the Yoneda embedding $\mathrm{y}\colon X \to \max(BX)$ is a homeomorphism from the natural topology on X generated by sets of the form

$$O(x, r) := \{y \in X \mid r \prec \Downarrow(x, y)\}$$

to the subspace Scott topology on BX.

In fact,

$$O(x, r) = \mathrm{y}^{-1}(\Uparrow\langle x, r\rangle).$$

2. On a duality theorem

Let X be $\mathbb{A}$-cocomplete and $\mathbb{A}$-continuous.

- For every $\mathcal{Q}$-functor $f\colon X \to Y$
  we have a $\mathcal{Q}$-functor $\underline{f}\colon \mathbb{A}X \to \mathbb{A}Y$:

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- Define
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  $$\mathcal{F}(f)(\alpha) := \alpha \circ f.$$

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- For $\mathcal{Q} = \mathbf{2}$, the above is known as the Lawson duality.
- The duality has been extended to J-continuous J-cocomplete (T, V)-categories (with Dirk Hofmann).

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- We'll show more: both are instances of a single theorem with a constructive proof.

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7. If $z \in X$ is other fixed point, then $C \subseteq {\downarrow}z$, hence $\mathit{fix}(T) \leqslant z$.

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(Caristi, 1976) Let $T: X \to X$ be arbitrary map on a complete metric space. If there exists a l.s.c map $\varphi: X \to [0, \infty)$ such that:

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Proof: ϕ l.s.c. iff $Z := \{\langle x, \phi x\rangle \mid x \in X\} \subseteq \mathbf{B}X$ is a dcpo. Moreover, (*) iff $\langle x, \phi x\rangle \leqslant \langle Tx, \phi(Tx)\rangle$ in $\mathbf{B}X$. Hence (*) holds iff $T' : Z \to Z$, given by $T'(\langle x, \phi x\rangle) := \langle Tx, \phi(TX)\rangle$ is expanding. Hence (Nonsymmetric Caristi) is constructive iff the Bourbaki-Witt theorem is constructive.

Instead of conclusion

(Caristi, 1976) Let $T: X \to X$ be arbitrary map on a complete metric space. If there exists a l.s.c map $\varphi: X \to [0, \infty)$ such that:

$$(*) \qquad X(x, Tx) + \phi(Tx) \leqslant \phi(x),$$

then T has a fixed point.

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Proof: ϕ l.s.c. iff $Z := \{\langle x, \phi x\rangle \mid x \in X\} \subseteq \mathbf{B}X$ is a dcpo. Moreover, (*) iff $\langle x, \phi x\rangle \leqslant \langle Tx, \phi(Tx)\rangle$ in $\mathbf{B}X$. Hence (*) holds iff $T'\colon Z \to Z$, given by $T'(\langle x, \phi x\rangle) := \langle Tx, \phi(TX)\rangle$ is expanding. Hence (Nonsymmetric Caristi) is constructive iff the Bourbaki-Witt theorem is constructive. But Andrej Bauer proved that the Bourbaki-Witt theorem has no constructive proof. qed.