

Surviving without function types:

Life in an arithmetic universe

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Work in progress joint with Milly Maietti

- Arithmetic universes as generalized spaces
- An induction principle some categories of sheaves
- Classical logic — of subspaces

Part I

Arithmetic universes

Arithmetic universe

Originated: Joyal (1970s) unpublished

- constructed examples (inc. initial one)
- applied to Gödel's theorem

General definition?

Pretopos + internal free algebras ... which? Joyal,raith... free categories & diagrams

Maietti:

AU = list-arithmetic pretopos
pretopos + parametrized list objects

Cockett & Jay: algebra in locales

A category is a pretopos if

- has finite limits
- has finite coproducts
- and they are stable and disjoint
- has stable, effective quotients of equivalence relations

cf. Giraud's theorem

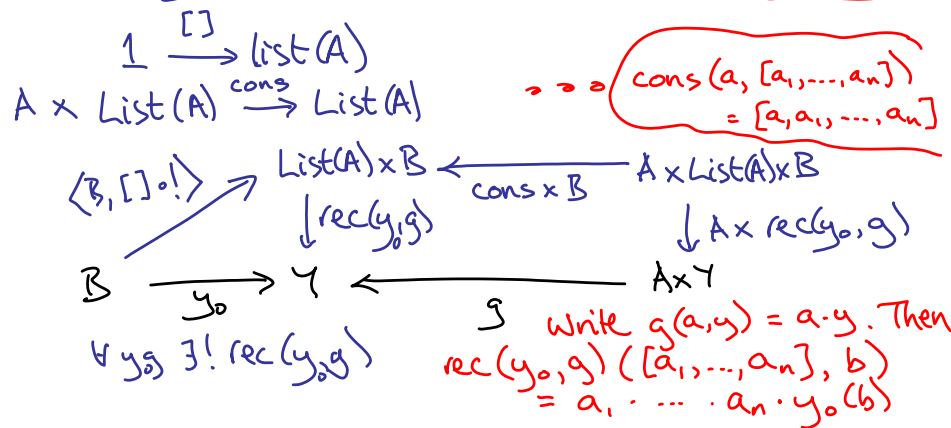
Similar conditions + all small coproducts + size constraints

⇒ Grothendieck topos			
• \mathbb{N} , hence free algebras	✓	✓	✗
• cartesian closed	✓	✗	✗
• power objects	✗	✗	✗

Arithmetic universe = list-arithmetic pretopos

For every A : $\text{List}(A)$ has

enough for
finitary algebra



$\text{cons}(a, [a_1, \dots, a_n]) = [a, a_1, \dots, a_n]$

Classifying AUs

Theory of AUs is cartesian

\Rightarrow can present AUs by generators & relations

Geometric theory

\downarrow
AU presentation

IF can replace V by internal types + \exists

e.g.

$$\begin{array}{c}
 V \\
 \downarrow \\
 \exists q: Q. L(q)
 \end{array}$$

AU stands in for classifying topos

AUs support limited fragment of sheaf theory
 how limited?

Arithmetic spaces

Generalized space = Grothendieck topos

Continuous Map = functor (backwards) of frames, locales

geometric morphism $\left[\begin{array}{l} \text{preserving all colimits} \\ \text{finite limits} \end{array} \right.$ has right adjoint
 also free algebras

Idea: Arithmetic space X given by AU $\mathcal{A}X$

Map $f: X \rightarrow Y$ is functor $\mathcal{A}X \leftarrow \mathcal{A}Y$

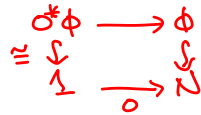
preserving finite colimits
 finite limits
 List strictly or non-strictly?

Part II

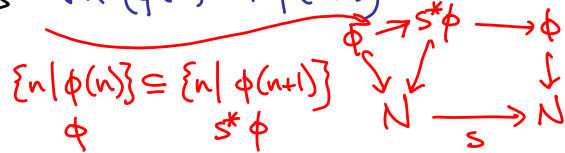
INDUCTION

Induction I $\phi \subseteq \mathbb{N}$. Suppose:

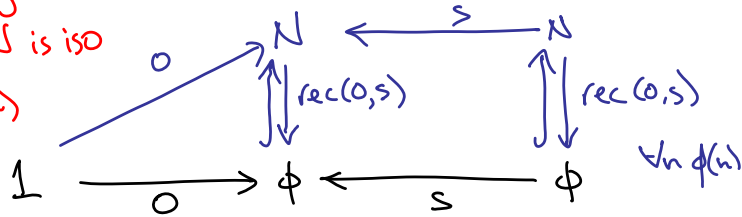
Base case $\phi(0)$



Induction step $\forall n (\phi(n) \rightarrow \phi(n+1))$



Can show
 $\phi \hookrightarrow \mathbb{N}$ is iso
i.e.
 $\forall n \phi(n)$



Induction for $\phi \rightarrow \psi$

Two predicates ϕ, ψ on \mathbb{N}

$\phi, \psi \subseteq \mathbb{N}$

Want $\forall n \phi(n) \rightarrow \psi(n)$

$\phi \subseteq \psi$

Function types give predicate $\phi \rightarrow \psi$

Use same induction:

Base case $\phi(0) \rightarrow \psi(0)$

Induction step $\forall n (\phi(n) \rightarrow \psi(n)) \rightarrow (\phi(n+1) \rightarrow \psi(n+1))$

$\forall n (\phi(n) \rightarrow \psi(n))$

Without function types?

\vdash classical logic —
 $\phi(n) \rightarrow \psi(n) \equiv \neg \phi(n) \vee \psi(n)$

$(\phi(n) \rightarrow \psi(n)) \rightarrow (\phi(n+1) \rightarrow \psi(n+1))$

$\equiv (\neg \phi(n) \vee \psi(n)) \rightarrow (\neg \phi(n+1) \vee \psi(n+1))$

$\equiv \neg \phi(n) \rightarrow (\neg \phi(n+1) \vee \psi(n+1))$

$\equiv \wedge \psi(n) \rightarrow (\neg \phi(n+1) \vee \psi(n+1))$

$\equiv \phi(n+1) \rightarrow \phi(n) \vee \psi(n+1)$

$\wedge \phi(n+1) \wedge \psi(n) \rightarrow \psi(n+1)$

} Know how to interpret these

Theorem (Marek/Nickers)

In any arithmetic universe: if have $\phi, \psi \subseteq \mathbb{N}$
 $\phi(0) \rightarrow \psi(0)$

$\forall n (\phi(n+1) \rightarrow \phi(n) \vee \psi(n+1))$

$\forall n (\phi(n+1) \wedge \psi(n) \rightarrow \psi(n+1))$

Then $\forall n (\phi(n) \rightarrow \psi(n))$

Proof

Define

$$(k: \mathbb{N}) A(k) \subseteq \mathbb{N}$$

$$A(k) = \{j \in \mathbb{N} \mid j \leq k, \phi(j), \dots, \psi(k)\}$$

Define $f_k : A(k) \rightarrow \psi(k)$ recursively so $f_k(j) = k$

$j = k = 0$: $\phi(0), \dots, \psi(0)$ by base case

$j = k > 0$: $\phi(j)$. By IS1, $\phi(j-1) \vee \psi(j)$
 $f_k(j) = \begin{cases} f_k(j-1) & \text{if } \phi(j-1) \\ k & \text{if } \psi(j) \end{cases}$

$j < k$: From $f_{k-1}(j)$ get $\psi(k-1)$
 $\therefore \phi(k) \wedge \psi(k-1)$, so $\psi(k)$ by IS2

In an arithmetic universe: if have $\phi, \psi \in \mathbb{N}$
 $\phi(0) \rightarrow \psi(0)$ - base case
 $\forall n (\phi(n+1) \rightarrow \phi(n) \vee \psi(n+1))$ IS1
 $\forall n (\phi(n+1) \wedge \psi(n) \rightarrow \psi(n+1))$ IS2
 Then $\forall n (\phi(n) \rightarrow \psi(n))$

If $\phi(k)$ then
 $k \in A(k)$ - use
 $f_k(k)$

recursion
 variant
 $= k+j$

Summary

\vdash classical logic in AU then
 induction proof for $\phi(n) \rightarrow \psi(n)$
 reduces to new induction principle

In an arithmetic universe: if have $\phi, \psi \in \mathbb{N}$
 $\phi(0) \rightarrow \psi(0)$
 $\forall n (\phi(n+1) \rightarrow \phi(n) \vee \psi(n+1))$
 $\forall n (\phi(n+1) \wedge \psi(n) \rightarrow \psi(n+1))$
 Then $\forall n (\phi(n) \rightarrow \psi(n))$

New principle valid in any AU

Part III

for any AU

Classical logic of subspaces

- Replace subsets of \mathbb{N} by subspaces
 open subspaces \parallel closed subspaces
 \Rightarrow Boolean complements
- Boolean calculations on subspaces
 \Rightarrow valid properties of subsets

X an AS: Subspace triple = $u \leftarrow \begin{matrix} \downarrow \\ S \\ \downarrow \\ v \end{matrix} \rightarrow v$ two subsheaves
 In topos: would use local operators
 sheaf $\dots =$ object of $\mathcal{A}X$

Corresponding subspace is $X[u \leq_I v] \hookrightarrow X$
 Preorder $u \leq_I v \iff u' \leq_{I'} v'$ if $\mathcal{A}X[u \leq_I v]$ also has $u' \leq_{I'} v'$

Meets $u_1 \leq_{I_1} v_1 \wedge u_2 \leq_{I_2} v_2 = u_1 + u_2 \leq_{I_1 + I_2} v_1 + v_2$

Special cases for $\phi, \psi \rightarrow 1$
 Open $\psi = \perp \leq_I \phi$ Crescent $(X - \phi) \wedge \psi$
 Closed $X - \phi = \phi \leq_I \perp$ Cocrescent $\phi \vee \psi$
 $(X - \phi) \vee \psi$



Representation theorems

Maretti: type theory

① Local homeomorphisms

$$\mathcal{A}X [a: 1 \rightarrow A] \simeq \mathcal{A}X/A$$

Adjoin generic element of A
 $A \rightrightarrows A \times A$
 \downarrow
 \mathcal{A}

Special case: opens $A = \phi \leftrightarrow 1$

$$\mathcal{A}\phi \simeq \mathcal{A}X/\phi$$

Corollary

$$\phi \simeq \begin{array}{c} u \\ \downarrow \downarrow \\ \Gamma \end{array} \begin{array}{c} v \\ \downarrow \downarrow \\ \Gamma \end{array} \iff u \times \phi \simeq_{\Gamma} v \times \phi$$

Representation theorems ② Closed $X - \phi$

Closed subspace of 1 is stone



$$B_\phi = 2 / (0=1 \text{ if } \phi)$$

Sheaves F over B_ϕ $F(0) = 1$
 $Sh(B_\phi)$ restriction $F(1) \rightarrow 1$ iso if ϕ

$$T_\phi: \mathcal{A}X \rightarrow \mathcal{A}X \quad \begin{array}{ccc} u \times \phi & \xrightarrow{u} & u \\ & \searrow \phi & \downarrow \\ & & u + \phi \end{array} \xrightarrow{\text{coequalizer}} T_\phi(u)$$

Thm: $sh(B_\phi) \simeq Alg(T_\phi) \simeq \mathcal{A}(X - \phi)$
 reflective subcat of $\mathcal{A}X$

Corollaries

open $\psi \simeq \begin{array}{c} u \\ \downarrow \downarrow \\ \Gamma \end{array} \begin{array}{c} v \\ \downarrow \downarrow \\ \Gamma \end{array} \iff u \times \psi \simeq_{\Gamma} v \times \psi$ in $\mathcal{A}X$ done already

closed $X - \phi \simeq \begin{array}{c} u \\ \downarrow \downarrow \\ \Gamma \end{array} \begin{array}{c} v \\ \downarrow \downarrow \\ \Gamma \end{array} \iff u \simeq_{\Gamma} v \vee \Gamma \times \phi$ in $\mathcal{A}X$

meets of opens
 subspace meet $\phi_1 \wedge \phi_2 = \text{subspace for subobject meet } \phi_1 \wedge \phi_2$
 $\phi_1 \wedge \phi_2 = \phi_1 \wedge \phi_2$

meets of closed
 $(X - \phi_1) \wedge (X - \phi_2) = X - (\phi_1 \vee \phi_2)$

crescent
 $(X - \phi) \wedge \psi \simeq \begin{array}{c} u \\ \downarrow \downarrow \\ \Gamma \end{array} \begin{array}{c} v \\ \downarrow \downarrow \\ \Gamma \end{array} \iff u \times \psi \simeq_{\Gamma} v \vee \Gamma \times \phi$ in $\mathcal{A}X$

Joins

Subspace for subobject join $\psi_1 \vee \psi_2$ is subspace join

$$\psi_1 \vee \psi_2 = \psi_1 \vee \psi_2$$

$X - (\phi_1 \wedge \phi_2)$ is subspace join $(X - \phi_1) \vee (X - \phi_2)$

corescent
 $\phi \downarrow \begin{array}{c} \psi \\ \downarrow \downarrow \\ \Gamma \end{array} \iff X - \phi, \psi \text{ both } \simeq \begin{array}{c} u \\ \downarrow \downarrow \\ \Gamma \end{array} \begin{array}{c} v \\ \downarrow \downarrow \\ \Gamma \end{array}$
 \Rightarrow obvious
 $\Leftarrow u \simeq v \vee u \times \phi \simeq v \vee u \times \psi \simeq v$

$\phi \downarrow \begin{array}{c} \psi \\ \downarrow \downarrow \\ \Gamma \end{array} = (X - \phi) \vee \psi$ cocrescents as joins
 Also: $(X - \phi) \vee \phi = X$ open & closed are Boolean complements
 $(X - \phi) \wedge \phi = 0$

Lattice structure

γ any subspace of X
 ϕ_i, ψ_i opens

Lemma (A) $\bigvee_{i=1}^n (\gamma \wedge (X - \phi_i) \wedge \psi_i)$ exists, and equals

(B) $\gamma \wedge \bigwedge \{ X [\bigwedge_{i \in J} \phi_i \leq \bigvee_{i \in K} \psi_i] \mid \{1, \dots, n\} = J \cup K \}$

Proof (1) Conjuncts in (B) are upper bounds for disjuncts in (A) both finite

(2) Over γ , suppose $\forall i \gamma \wedge (X - \phi_i) \wedge \psi_i \leq \bigvee_{i \in I} \psi_i$
 & adjoin all $\bigwedge_{i \in J} \phi_i \leq \bigvee_{i \in K} \psi_i$. Prove by induction $n - |J|$
 $U \wedge \bigwedge_{i \in J} \phi_i \leq V$

Corollary (1) Finite joins of crescents exist & are meets of cocrescents
 (2) Any γ distributes over those joins.

Lattice structure

Theorem

Finite joins of crescents
 = finite meets of cocrescents

They form a Boolean algebra.

Conjecture: have all finite joins of subspaces?

$$u_1 \vee_{I_1} v_1 \vee_{I_2} u_2 \vee_{I_2} v_2 = u_1 \wedge_{I_1} u_2 \vee_{I_1 \times I_2} v_1 \vee_{I_1 \times I_2} v_2 ?$$

Induction

$\phi, \psi \leftrightarrow N$ in $\mathcal{A}X$
 $\phi^{(n)}, \psi^{(n)} \leftrightarrow 1$ in $\mathcal{A}X[n:1 \rightarrow N]$
 $\phi^{(n+1)}, \psi^{(n+1)}$

Theorem Suppose $\phi^{(0)} \leq \psi^{(0)}$ in $\mathcal{A}X$
 $\phi^{(n)}, \psi^{(n)} \leq \phi^{(n+1)}, \psi^{(n+1)}$ over $\mathcal{A}X[n]$

Then $\phi \leq \psi$ in $\mathcal{A}X$

Proof $(X[n] - \phi^{(n)}) \vee \psi^{(n)} \leq (X[n] - \phi^{(n+1)}) \vee \psi^{(n+1)}$
 i.e. $\phi^{(n+1)} \leq \phi^{(n)} \vee \psi^{(n+1)}$ over $X[n]$

$\phi^{(n+1)} \wedge \psi^{(n)} \leq \psi^{(n+1)}$
 From $\mathcal{A}X[n] \simeq \mathcal{A}X/N$ deduce corresponding relations in $\mathcal{A}X$ \therefore can use previous induction thm

Conclusions

Classical logic of (some) subspaces
 even when logic of subobjects not classical

\Rightarrow can work with implications
 even when no internal exponentials

General moral Good properties of spaces spoiled
 when you discretize (take set of points)

e.g. Closed complement properly a Stone space
 - don't expect a subobject of 1