# Relative commutator theory and the associator of loops

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(joint work with Tomas Everaert)

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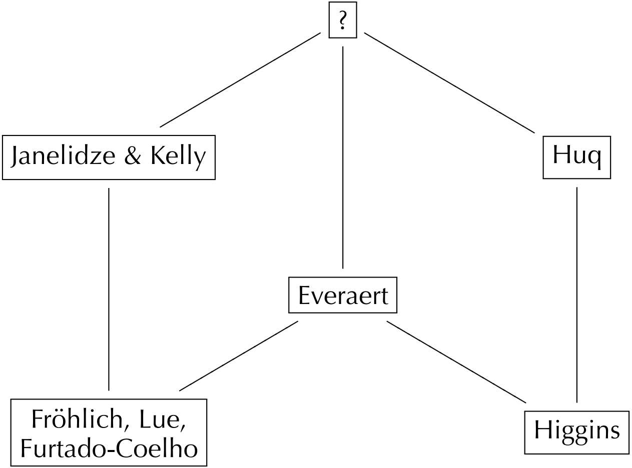
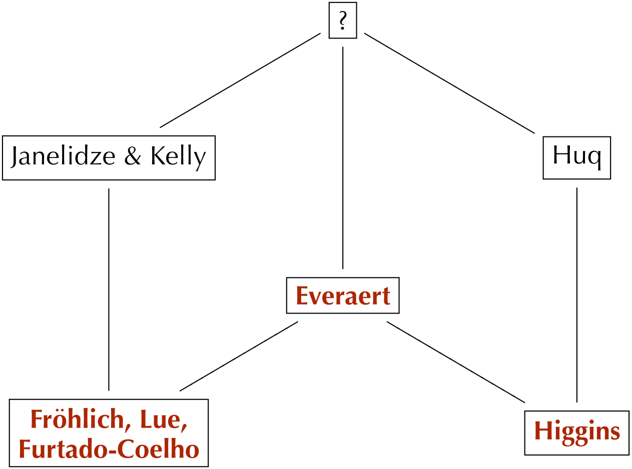
Genova, 21st of June 2010

# Some commutator theories,
# notions of pairs of commuting subobjects



# The case of Ω-groups



# The case of $\Omega$-groups

**Definition (Higgins)**

A *variety of $\Omega$-groups* is a pointed variety, with operations and identities containing those of **Gp**

- examples: all varieties of groups, rings, modules; all kinds of algebras over rings; precrossed and crossed modules
- if $M, N \triangleleft A$ then $M \vee N = M \cup N = M \cdot N$
- any subvariety $\mathcal{B} \leq \mathcal{A}$ given by set of identities $w(\boldsymbol{x}) = 1$
- this determines an $\Omega$-group

$$W_{\mathcal{B}} = \{w \text{ term} \mid w(\boldsymbol{b}) = 1, \forall B \in \mathcal{B}, \forall \boldsymbol{b} \in B\}$$

- $A \in \mathcal{B}$ iff $w(\boldsymbol{a}) = 1$ for every term $w \in W_{\mathcal{B}}$ and every $\boldsymbol{a} \in A$

# The case of Ω-groups

**The Higgins commutator:** *absolute, two-dimensional*

For $M$, $N \lhd A$ the commutator $[M,N]^{\mathrm{H}} \lhd M \vee N$ is generated by

$$\{w(\boldsymbol{mn})w(\boldsymbol{n})^{-1}w(\boldsymbol{m})^{-1} \mid w \text{ term}, \boldsymbol{m} \in M \text{ and } \boldsymbol{n} \in N\}$$

Let $\mathcal{B}$ be a subvariety of $\mathcal{A}$

**The Fröhlich commutator:** *relative, one-dimensional*

For $K \lhd A$ the commutator $[K,A]^{\Omega}_{\mathcal{B}} \lhd A$ is generated by

$$\{w(\boldsymbol{ka})w(\boldsymbol{a})^{-1} \mid w \in W_{\mathcal{B}}, \boldsymbol{k} \in K \text{ and } \boldsymbol{a} \in A\}$$

**The Everaert commutator:** *relative, two-dimensional*

For $M$, $N \lhd A$ the commutator $[M,N]^{\Omega}_{\mathcal{B}} \lhd M \vee N$ is generated by

$$\{w(\boldsymbol{mn})w(\boldsymbol{n})^{-1}w(\boldsymbol{m})^{-1}w(\boldsymbol{p}) \mid w \in W_{\mathcal{B}}, \boldsymbol{m} \in M, \boldsymbol{n} \in N, \boldsymbol{p} \in M \wedge N\}$$

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# The case of Ω-groups

- Higgins characterises $\mathbf{Ab}(\mathcal{A})$: $A$ is abelian iff $[A, A]^{\mathrm{H}} = 0$

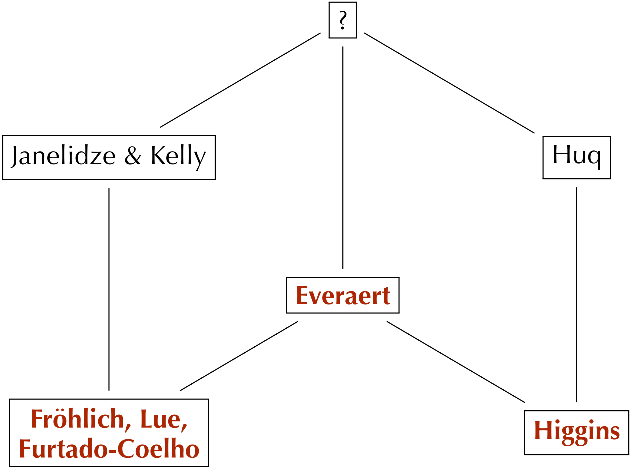
  Examples: usual commutator for **Gp**, **Rng**, $\mathbf{Lie}_K$, etc.
- Fröhlich's central extensions: an extension $f\colon A \to B$ with kernel $K$ is $\mathcal{B}$-central iff $[K, A]^{\Omega}_{\mathcal{B}} = 0$

  Fröhlich characterises $\mathcal{B}$: $A$ is in $\mathcal{B}$ iff $[A, A]^{\Omega}_{\mathcal{B}} = 0$

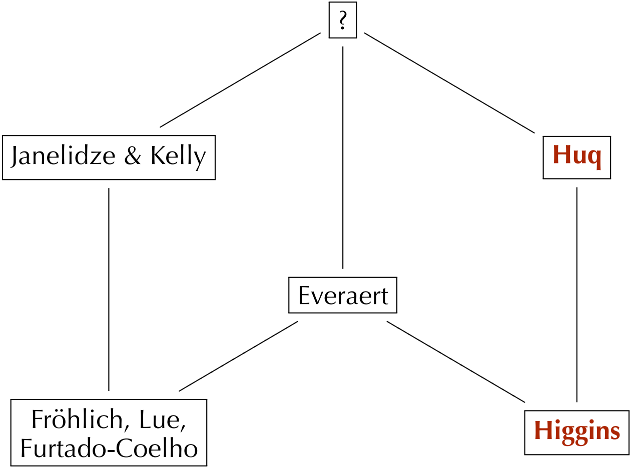
  Examples: central extensions for **Leib** vs. **Lie**, **PXMod** vs. **XMod**, etc.
- In the one-dimensional absolute case, $[K, A]^{\Omega}_{\mathbf{Ab}(\mathcal{A})} = [K, A]^{\mathrm{H}}$
- Everaert's commutator generalises both of them

  Example: **PXMod** vs. **XMod** gives the Peiffer commutator

# The case of Ω-groups



# The *two-dimensional* and *absolute* theories



# The *two-dimensional* and *absolute* theories

## The Huq commutator: categorical version of Higgins

$\mathcal{A}$ semi-abelian

For $M$, $N \lhd A$ the commutator $[M, N]^{\mathrm{Q}}$ is the kernel of $q$:

$$\begin{tikzcd}
 & M \arrow[dl, "\langle 1_M{,}0\rangle"'] \arrow[d, dotted] \arrow[dr, "m"] & \\
M \times N \arrow[r, dotted] & Q & A \arrow[l, dotted, "q"'] \\
 & N \arrow[ul, "\langle 0{,}1_N\rangle"] \arrow[u, dotted] \arrow[ur, "n"'] &
\end{tikzcd}$$

- Further studied by Borceux, Bourn & Gran
- $A \in \mathcal{A}$ is abelian iff $[A, A]^{\mathrm{Q}} = 0$
- If $\mathcal{A}$ is a variety of $\Omega$-groups, $[M, N]^{\mathrm{Q}} = [M, N]^{\mathrm{H}}$ for $[M, N]^{\mathrm{Q}}$ calculated in $M \vee N$
- Alternative approach: Higgins commutator of Mantovani & Metere, Hartl & Loiseau

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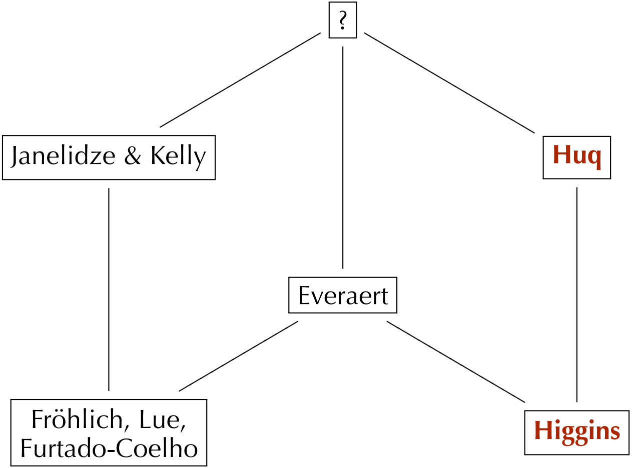

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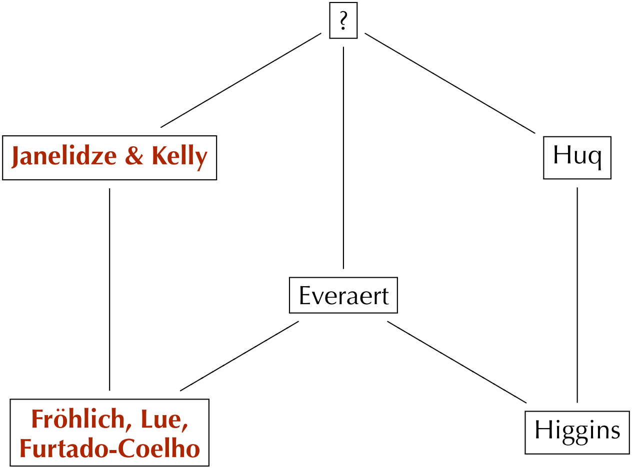

# The *one-dimensional* and *relative* theories

- In a semi-abelian category $\mathcal{A}$, an *extension* is a regular epimorphism
- $\mathcal{B}$ Birkhoff subcategory of $\mathcal{A}$ gives adjunction $\begin{tikzcd} \mathcal{A} \arrow[r, shift left, "I"] \arrow[r, phantom, "\perp"] & \mathcal{B} \arrow[l, shift left, "\supset"] \end{tikzcd}$

## Trivial and central extensions (Janelidze & Kelly)

- $f\colon A \to B$ is $\mathcal{B}$-*trivial* iff $\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "\eta_A"'] & B \arrow[d, "\eta_B"] \\ IA \arrow[r, "If"'] & IB \end{tikzcd}$ pullback
- $f\colon A \to B$ is $\mathcal{B}$-*central* iff $f_0\colon R[f] \to A$ is $\mathcal{B}$-trivial

- Categorical Galois theory (Janelidze)
- Gives Fröhlich's definition in varieties of $\Omega$-groups

# The *one-dimensional* and *relative* theories

$$
\begin{tikzcd}
 & {[K,A]_{\mathcal{B}}} \arrow[d, "{\ker[f_0]_{\mathcal{B}}}"'] \arrow[drr, dotted] & & & \\
0 \arrow[r] & {[R[f]]_{\mathcal{B}}} \arrow[r, "\mu_{R[f]}"] \arrow[d, shift right, "{[f_0]_{\mathcal{B}}}"'] \arrow[d, shift left, "{[f_1]_{\mathcal{B}}}"] & R[f] \arrow[r, "\eta_{R[f]}"] \arrow[d, shift right, "f_0"'] \arrow[d, shift left, "f_1"] & IR[f] \arrow[r] \arrow[d, shift right, "If_0"'] \arrow[d, shift left, "If_1"] & 0 \\
0 \arrow[r] & {[A]_{\mathcal{B}}} \arrow[r, "\mu_A"'] & A \arrow[r, "\eta_A"'] & IA \arrow[r] & 0
\end{tikzcd}
$$

- $f\colon A \to B$ with kernel $K$ is $\mathcal{B}$-central
  iff $[f_0]_{\mathcal{B}}$ iso iff $[f_0]_{\mathcal{B}} = [f_1]_{\mathcal{B}}$ iff $[K,A]_{\mathcal{B}} = 0$
- Gives adjunction $\begin{tikzcd} \mathbf{Ext}(\mathcal{A}) \arrow[r, shift left, "I_1"] \arrow[r, phantom, "\perp"] & \mathbf{CExt}_{\mathcal{B}}(\mathcal{A}) \arrow[l, shift left, "\supset"] \end{tikzcd}$
- $A \in \mathcal{B}$ iff $[A,A]_{\mathcal{B}} = 0$
- $[K,A]_{\mathcal{B}} = [K,A]^{\Omega}_{\mathcal{B}}$ in the case of varieties of $\Omega$-groups

# The *one-dimensional* and *relative* theories

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\begin{tikzcd}
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- $A \in \mathcal{B}$ iff $[A,A]_{\mathcal{B}} = 0$
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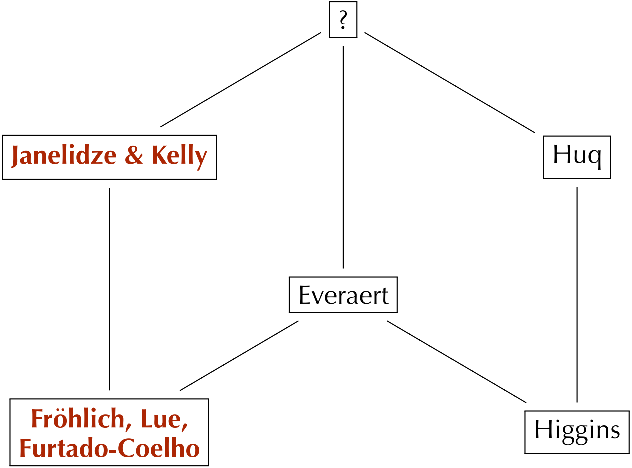
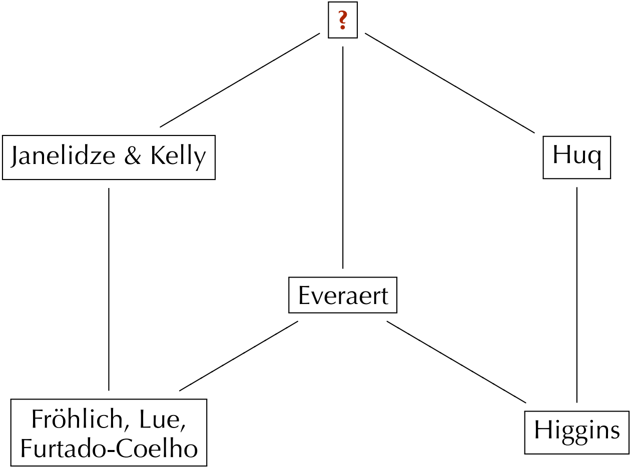
# The *one-dimensional* and *relative* theories



# A categorical, *two-dimensional* and *relative* theory which generalises the case of Ω-groups



# The fundamental theorem

**Theorem**

*Let $\mathcal{B} \leq \mathcal{A}$ be varieties of $\Omega$-groups*

*For $M, N \lhd A \in \mathcal{A}$ the commutator $[M, N]^{\Omega}_{\mathcal{B}}$ is zero iff the right hand square is a double $\mathcal{B}$-central extension*

$$\begin{tikzcd}
M \vee N \arrow[r, "q_M"] \arrow[d, "q_N"'] & \frac{M \vee N}{M} \arrow[d] \\
\frac{M \vee N}{N} \arrow[r] & 0
\end{tikzcd}$$

- Conceptual explanation for Everaert's definition

**General definition**

$\mathcal{B} \leq \mathcal{A}$ semi-abelian

$M, N \lhd A$ normal subobjects $\mathcal{B}$-*commute* iff the square is a double $\mathcal{B}$-central extension

- New examples: **Loop** vs. **Gp**, etc.

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$$\begin{array}{ccc} M \vee N & \xrightarrow{q_M} & \frac{M \vee N}{M} \\ {\scriptstyle q_N}\downarrow & & \downarrow \\ \frac{M \vee N}{N} & \longrightarrow & 0 \end{array}$$

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# Double extensions

A *double extension* is a commutative square such that every arrow in the induced diagram is an extension.

$$\begin{tikzcd} X \arrow[r, "c"] \arrow[d, "d"'] & C \arrow[d, "g"] \\ D \arrow[r, "f"'] & Z \end{tikzcd}$$

For any $M, N \lhd A$, $\begin{tikzcd} M \vee N \arrow[r, "q_M"] \arrow[d, "q_N"'] & \frac{M \vee N}{M} \arrow[d] \\ \frac{M \vee N}{N} \arrow[r] & 0 \end{tikzcd}$ is a double extension

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# Double central extensions

**Definition**

A *double central extension* is a double extension (in particular, a morphism in $\mathbf{Ext}(\mathcal{A})$) which is central w.r.t. the adjunction

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- concept due to R. Brown,
- worked out for groups vs. abelian groups by Janelidze
- higher-dimensional, general version by Everaert, Gran & VdL
  semi-abelian homology, appears in Hopf formula for $H_3$
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# Definition of the commutator

- For $J \lhd M \vee N$, let $R_J$ be the kernel pair of $q_J \colon M \vee N \to \frac{M \vee N}{J}$

## Proposition

$$\begin{tikzcd} M \vee N \arrow[r, "q_M"] \arrow[d, "q_N"'] & \frac{M \vee N}{M} \arrow[d] \\ \frac{M \vee N}{N} \arrow[r] & 0 \end{tikzcd}$$ central iff $$\begin{tikzcd} {[R_M \square R_N]_{\mathcal{B}}} \arrow[r, shift left, "{[r_1]_{\mathcal{B}}}"] \arrow[r, shift right, "{[r_0]_{\mathcal{B}}}"'] \arrow[d, shift right, "{[p_0]_{\mathcal{B}}}"'] \arrow[d, shift left, "{[p_1]_{\mathcal{B}}}"] & {[R_N]_{\mathcal{B}}} \arrow[d, shift right] \arrow[d, shift left] \\ {[R_M]_{\mathcal{B}}} \arrow[r, shift left] \arrow[r, shift right] & {[M \vee N]_{\mathcal{B}}} \end{tikzcd}$$ pullbacks

## Definition

For $M$, $N \lhd A$ we write $[M, N]_{\mathcal{B}} = K[[p_0]_{\mathcal{B}}] \wedge K[[r_0]_{\mathcal{B}}]$

- $M$, $N \lhd A$ commute iff $[M, N]_{\mathcal{B}} = 0$
- Generalises Janelidze & Kelly's central extensions
- $[M, N]_{\mathbf{Ab}(\mathcal{A})} = [M, N]^{\mathrm{Q}}$ for $[M, N]^{\mathrm{Q}}$ computed in $M \vee N$ as "Smith is Huq" for $M$, $N \lhd A$ jointly strongly epic (Gran)

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# Basic stability properties of the commutator

**Proposition**

For $M$, $N$, $N' \lhd A$ and $J \lhd M \vee N$ the following hold:

- $[0, N]_{\mathcal{B}} = 0$;
- $[M, N]_{\mathcal{B}} = [N, M]_{\mathcal{B}}$;
- $[M, N]_{\mathcal{B}} \leq M \wedge N$;
- if $N \leq N'$ then $[M, N]_{\mathcal{B}} \leq [M, N']_{\mathcal{B}}$ as subobjects of $A$;
- $q_J[M, N]_{\mathcal{B}} \leq [q_J M, q_J N]_{\mathcal{B}}$;
- $q_J[M, N]_{\mathcal{B}} = [q_J M, q_J N]_{\mathcal{B}}$ as soon as $J \leq M \wedge N$;
- $[M, N]_{\mathcal{B}}$ is the smallest $J \lhd M \vee N$ such that $q_J M$ and $q_J N$ commute.

- The commutator does not preserve joins or direct images (Everaert, counterexamples in $\Omega$-groups)

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- The commutator does not preserve joins or direct images (Everaert, counterexamples in $\Omega$-groups)

# Basic stability properties of the commutator

**Proposition**

For $M, N, N' \lhd A$ and $J \lhd M \vee N$ the following hold:

- $[0, N]_{\mathcal{B}} = 0$;
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# The example of loops vs. groups

- A *loop* is an algebra $(A, \cdot, \backslash, /, 1)$ which satisfies $x \cdot 1 = x = 1 \cdot x$ and

$$y = x \cdot (x \backslash y) \qquad y = x \backslash (x \cdot y)$$
$$x = (x/y) \cdot y \qquad x = (x \cdot y)/y$$

- semi-abelian variety: $n = 1$, $t(x, y) = x \cdot y$, $t_1(x, y) = x/y$
- associative loop = group; this gives $\mathbf{Loop} \underset{\supset}{\overset{\mathrm{gp}}{\underset{\longleftarrow}{\overset{\longrightarrow}{\perp}}}} \mathbf{Gp}$
- associator: $[x, y, z] = ((x \cdot y) \cdot z)/(x \cdot (y \cdot z))$ for $x$, $y$, $z \in A$
- Consider $L$, $M$, $N \triangleleft A$
  $[L, M, N] \triangleleft L \vee M \vee N$ generated by $[l, m, n]$, $[m, n, l]$, etc.

# The example of loops vs. groups

For loops vs. groups, the relative commutator is an associator

**Theorem**

- $[A]_{\mathbf{Gp}} = [A, A, A]$
- for $K \lhd A$, $[K, A]_{\mathbf{Gp}} = [K, A, A]$
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This gives, for instance, the Hopf formula:

$$H_2(A, \mathbf{Gp}) \cong \frac{K \wedge [P, P, P]}{[K, P, P]}$$

for $A = P/K$ with $P$ projective.

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- it suffices that $f\colon A \to B$ with kernel $K$ is central iff $[K, A, A] = 0$
- $f$ is central iff $[f_0]_{\mathbf{Gp}} = [f_1]_{\mathbf{Gp}} \colon [R[f], R[f], R[f]] \to [A, A, A]$

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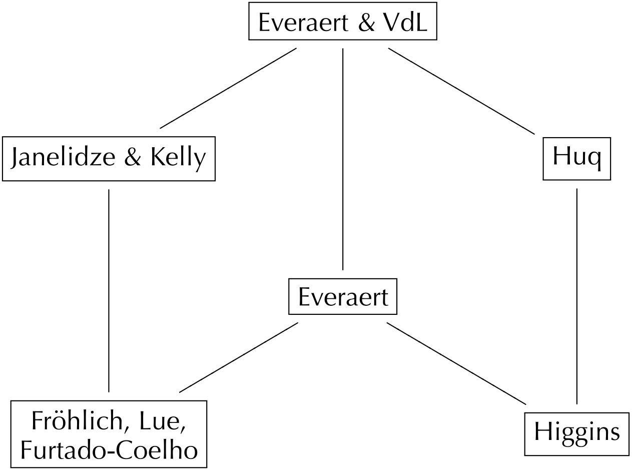
- Begs for a three-dimensional theory = three-fold central extensions
- Loops vs. commutative loops?
- Intrinsic associativity?


Everaert & VdL
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