The Web Monoid and Opetopic Sets

Stanisław Szawiel
joint work with
Marek Zawadowski

Institute of Mathematics
University of Warsaw

CT2010
21st June 2010

Goals

- Make M. Makkai's definition of weak ω-categories useable

Goals

- Make M. Makkai's definition of weak ω-categories useable
- "there is an obvious weak ω-category of ..."

Goals

- Make M. Makkai's definition of weak ω-categories useable
- "there is an obvious weak ω-category of ..."
- First step: make opetopic sets easier to work with

To build an opetopic set . . .

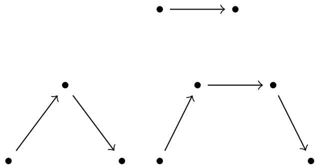
- . . . start with n-cells

Example ($n = 1$)

$$\bullet \longrightarrow \bullet$$

To build an opetopic set . . .

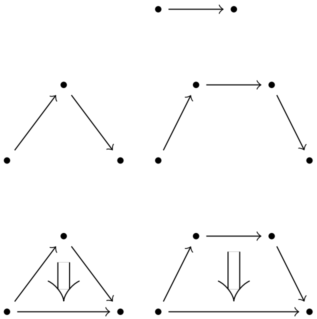
Example $(n = 1)$



- . . . start with n-cells
- Hocus-Pocus – determine the possible $(n + 1)$-cells

To build an opetopic set . . .

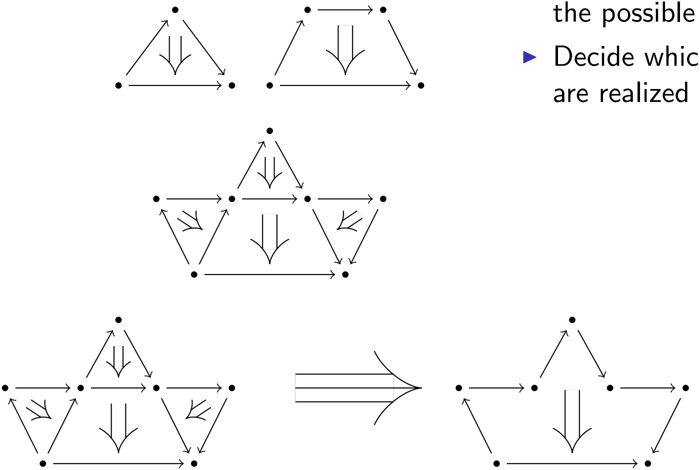
Example ($n = 1$)



- . . . start with n-cells
- Hocus-Pocus – determine the possible $(n + 1)$-cells
- Decide which $(n + 1)$-cells are realized

To build an opetopic set . . .

Example ($n = 2$)



- . . . start with n-cells
- Hocus-Pocus – determine the possible $(n+1)$-cells
- Decide which $(n+1)$-cells are realized

How to do magic

Question: how do we determine the possible $(n+1)$-cells?

How to do magic

Question: how do we determine the possible $(n+1)$-cells?
Answer: use a mysterious monoid

How to do magic

Question: how do we determine the possible $(n+1)$-cells?
Answer: use a mysterious monoid

- Operad for operads (Baez-Dolan). Motivated conceptually.

How to do magic

Question: how do we determine the possible $(n+1)$-cells?
Answer: use a mysterious monoid

- Operad for operads (Baez-Dolan). Motivated conceptually.
- Multicategory of function replacement (Hermida-Makkai-Power). Motivated geometrically.

How to do magic

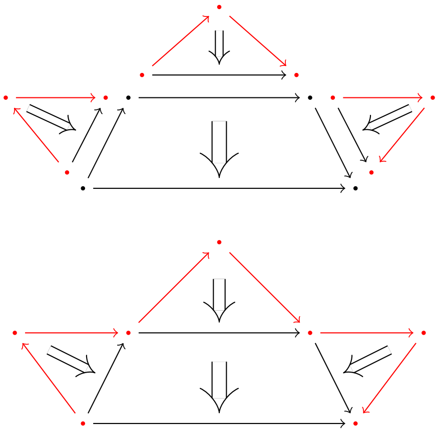
Question: how do we determine the possible $(n+1)$-cells?
Answer: use a mysterious monoid

- Operad for operads (Baez-Dolan). Motivated conceptually.
- Multicategory of function replacement (Hermida-Makkai-Power). Motivated geometrically.
- Web Monoid – seeks the middle ground.

The Geometrical Problem

following Hermida-Makkai-Power

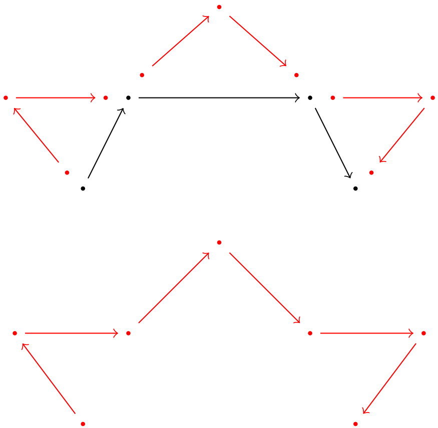
Problem: Composing cells pastes their domains!



The Geometrical Problem

following Hermida-Makkai-Power

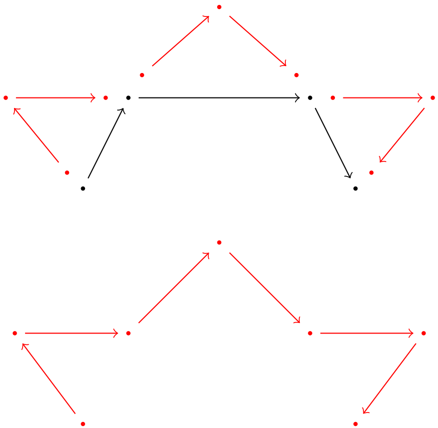
Question: what kind of operation is this?



The Geometrical Problem

following Hermida-Makkai-Power

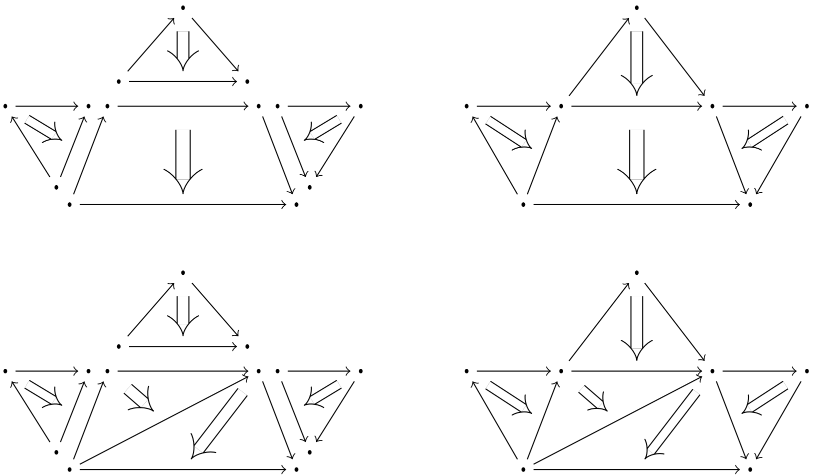
Question: what kind of operation is this?



Replace instances of cells (black) with formal composites of cells.

An Insight

This operation commutes with taking formal composites



Lax Monoidal Fibrations

Definition

A lax monoidal fibration over $\mathcal{B}$ is a lax monoid object in the 2-category of fibrations over $\mathcal{B}$, fibered functors and fibered natural transformations.

Lax Monoidal Fibrations

Definition

A lax monoidal fibration over $\mathcal{B}$ is a lax monoid object in the 2-category of fibrations over $\mathcal{B}$, fibered functors and fibered natural transformations.

In practice

A lax monoidal fibration over $\mathcal{B}$ is a fibration $\mathcal{E} \to \mathcal{B}$ and

Lax Monoidal Fibrations

Definition

A lax monoidal fibration over $\mathcal{B}$ is a lax monoid object in the 2-category of fibrations over $\mathcal{B}$, fibered functors and fibered natural transformations.

In practice

A lax monoidal fibration over $\mathcal{B}$ is a fibration $\mathcal{E} \to \mathcal{B}$ and

- The fibers are lax monoidal

Lax Monoidal Fibrations

Definition

A lax monoidal fibration over $\mathcal{B}$ is a lax monoid object in the 2-category of fibrations over $\mathcal{B}$, fibered functors and fibered natural transformations.

In practice

A lax monoidal fibration over $\mathcal{B}$ is a fibration $\mathcal{E} \to \mathcal{B}$ and

- The fibers are lax monoidal
- The reindexing functors are lax monoidal

Lax Monoidal Fibrations

Definition

A lax monoidal fibration over $\mathcal{B}$ is a lax monoid object in the 2-category of fibrations over $\mathcal{B}$, fibered functors and fibered natural transformations.

In practice

A lax monoidal fibration over $\mathcal{B}$ is a fibration $\mathcal{E} \to \mathcal{B}$ and

- The fibers are lax monoidal
- The reindexing functors are lax monoidal
- Our examples: fibers are strong, reindexing – never

Ordinary Signatures

with amalgamation

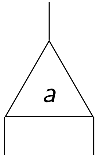
Category $\mathbf{Sig}_a$

Objects

Sets A together with a typing $\partial^A : A \to O \times O^*$

Ordinary Signatures

with amalgamation



Category $\mathbf{Sig}_a$

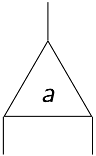
Objects

Sets A together with a typing $\partial^A : A \to O \times O^*$

- $a \in A$ are function symbols

Ordinary Signatures

with amalgamation



Category $\mathbf{Sig}_a$

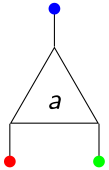
Objects

Sets A together with a typing $\partial^A : A \to O \times O^*$

- $a \in A$ are function symbols
- O is the set of *types* or *sorts*

Ordinary Signatures

with amalgamation



Category $\mathbf{Sig}_a$

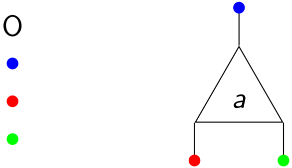
Objects

Sets A together with a typing $\partial^A : A \to O \times O^*$

- $a \in A$ are function symbols
- O is the set of *types* or *sorts*
- $\partial(a)$ lists output and input types of a

Ordinary Signatures

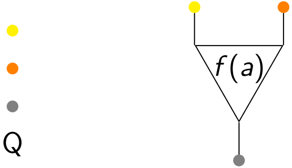
with amalgamation



Category $\mathbf{Sig}_a$

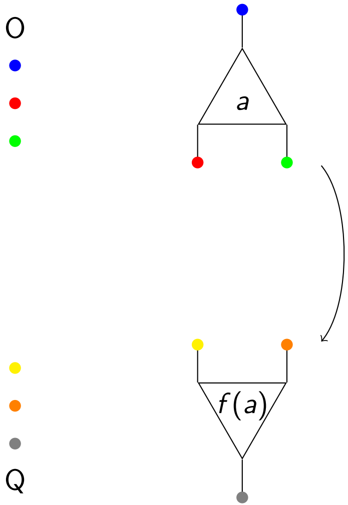
Morphisms

Triples (f, u, σ):



Ordinary Signatures

with amalgamation



Category $\mathbf{Sig}_a$

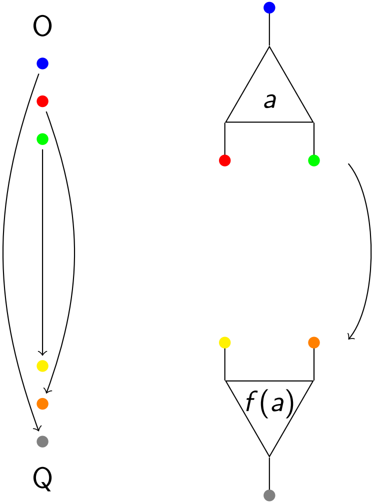
Morphisms

Triples (f, u, σ):

- $f : A \rightarrow B$ maps function symbols

Ordinary Signatures

with amalgamation



Category $\mathbf{Sig}_a$

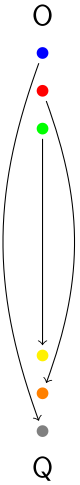
Morphisms

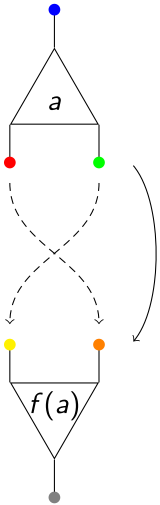
Triples (f, u, σ):

- $f : A \to B$ maps function symbols
- $u : O \to Q$ relates the types

Ordinary Signatures

with amalgamation





Category **Sig**$_a$

Morphisms

Triples (f, u, σ):

- $f : A \to B$ maps function symbols
- $u : O \to Q$ relates the types
- σ_a connects inputs of a to those of $f(a)$, respecting types

Lax Monoidal Fibration Structure

The functor (signature $\mapsto$ its types) defines the fibration $\mathbf{Sig}_a \to \mathbf{Set}$.

Lax Monoidal Fibration Structure

The functor (signature $\mapsto$ its types) defines the fibration $\mathbf{Sig}_a \to \mathbf{Set}$.

$A \otimes_O B$ consists of formal composites:

$$\{a(b_1, \ldots, b_k) | a \in A, b_i \in B, \text{inputs of } a = \text{outputs of } b_i\}$$

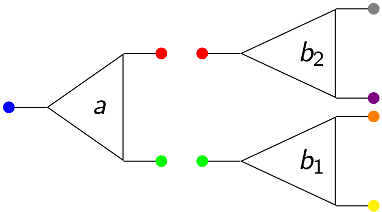
Lax Monoidal Fibration Structure

The functor (signature $\mapsto$ its types) defines the fibration $\mathbf{Sig}_a \to \mathbf{Set}$.

$A \otimes_O B$ consists of formal composites:

$$\{a(b_1, \ldots, b_k) | a \in A, b_i \in B, \text{inputs of } a = \text{outputs of } b_i\}$$

Typical element of $A \otimes_O B$:



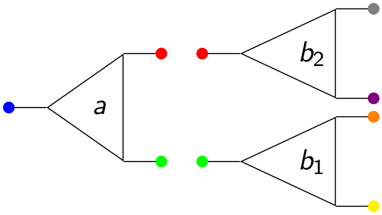
Lax Monoidal Fibration Structure

The functor (signature $\mapsto$ its types) defines the fibration $\mathbf{Sig}_a \to \mathbf{Set}$.

$A \otimes_O B$ consists of formal composites:

$$\{a(b_1, \ldots, b_k) | a \in A, b_i \in B, \text{inputs of } a = \text{outputs of } b_i\}$$

Typical element of $A \otimes_O B$:



Note the obvious typing

Monoidal Signatures

Set of types becomes a multicategory $M \in Mon(\mathbf{Sig}_a)$.

Monoidal Signatures

Set of types becomes a multicategory $M \in Mon(\mathbf{Sig}_a)$.

Objects of $\mathbf{Sig}_{ma}$

Objects over M are sets A with typing functions $A \to M \times M^*$.

Monoidal Signatures

Set of types becomes a multicategory $M \in Mon(\mathbf{Sig}_a)$.

Objects of $\mathbf{Sig}_{ma}$

Objects over M are sets A with typing functions $A \to M \times M^*$.

New typing appears

- $A \to M \times M^* \xrightarrow{\text{output type}} M \xrightarrow{\text{typing of M}} O \times O^*$

Monoidal Signatures

Set of types becomes a multicategory $M \in Mon(\mathbf{Sig}_a)$.

Objects of $\mathbf{Sig}_{ma}$

Objects over M are sets A with typing functions $A \to M \times M^*$.

New typing appears

- $A \to M \times M^* \xrightarrow{\text{output type}} M \xrightarrow{\text{typing of M}} O \times O^*$
- Horizontal typing.

Monoidal Signatures

Set of types becomes a multicategory $M \in Mon(\mathbf{Sig}_a)$.

Objects of $\mathbf{Sig}_{ma}$

Objects over M are sets A with typing functions $A \to M \times M^*$.

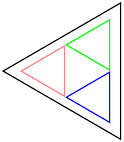
New typing appears

- $A \to M \times M^* \xrightarrow{\text{output type}} M \xrightarrow{\text{typing of M}} O \times O^*$
- Horizontal typing.

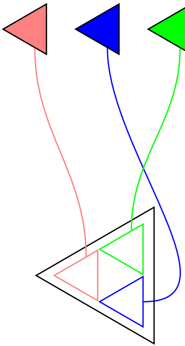
Morphisms

As in $\mathbf{Sig}_a$

Two Monoidal Structures

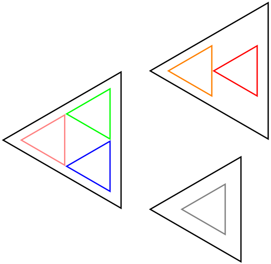


Two Monoidal Structures



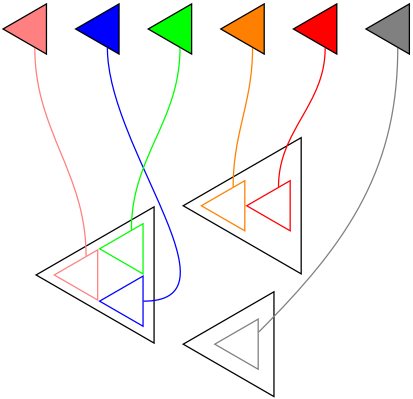
- Obvious one, $A \otimes_M B$, defined using typing in M

Two Monoidal Structures



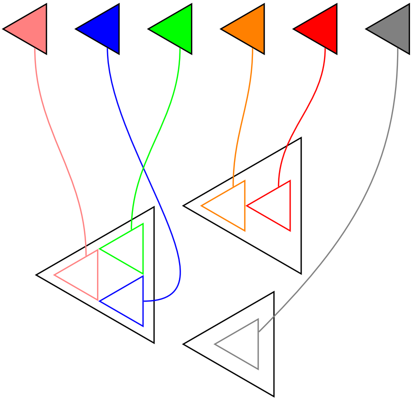
- New one: $A \odot_M B = A \otimes_O B$, defined by horizontal typing

Distributivity



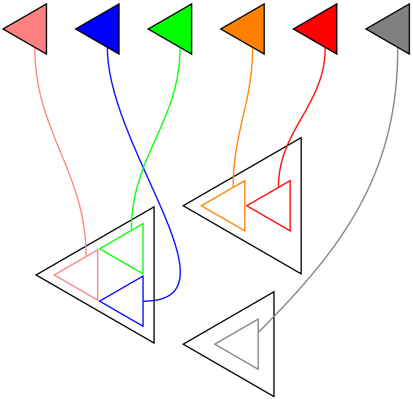
- An element of $(A \odot B) \otimes C$

Distributivity



- An element of $(A \odot B) \otimes C$
- Or maybe of $(A \otimes C) \odot (B \otimes C)$?

Distributivity



- An element of $(A \odot B) \otimes C$
- Or maybe of $(A \otimes C) \odot (B \otimes C)$?
- A natural isomorphism makes the picture unambiguous

The Three Tensors Theorem

distributivity of monoidal structures

- $\odot, \otimes$ - two (strong) monoidal structures on $\mathcal{C}$

The Three Tensors Theorem

distributivity of monoidal structures

- $\odot, \otimes$ - two (strong) monoidal structures on $\mathcal{C}$
- $\otimes$ distributes over $\odot$ if we are given natural isomorphisms:

$$\varphi_{A,B,C} : (A \otimes C) \odot (B \otimes C) \to (A \odot B) \otimes C$$

$$\dot{\varphi}_C : I_\odot \to I_\odot \otimes C$$

The Three Tensors Theorem

distributivity of monoidal structures

- $\odot, \otimes$ - two (strong) monoidal structures on $\mathcal{C}$
- $\otimes$ distributes over $\odot$ if we are given natural isomorphisms:

$$\varphi_{A,B,C} : (A \otimes C) \odot (B \otimes C) \to (A \odot B) \otimes C$$

$$\dot{\varphi}_C : I_\odot \to I_\odot \otimes C$$

which satisfy some coherence conditions

The Three Tensors Theorem

assumptions

- $\mathcal{C}$ - cocomplete category

The Three Tensors Theorem

assumptions

- $\mathcal{C}$ - cocomplete category
- $\coprod$, $\odot$, $\otimes$ - three finitary monoidal structures

The Three Tensors Theorem

assumptions

- $\mathcal{C}$ - cocomplete category
- $\coprod$, $\odot$, $\otimes$ - three finitary monoidal structures
- Each distributes over previous one

The Three Tensors Theorem

statement

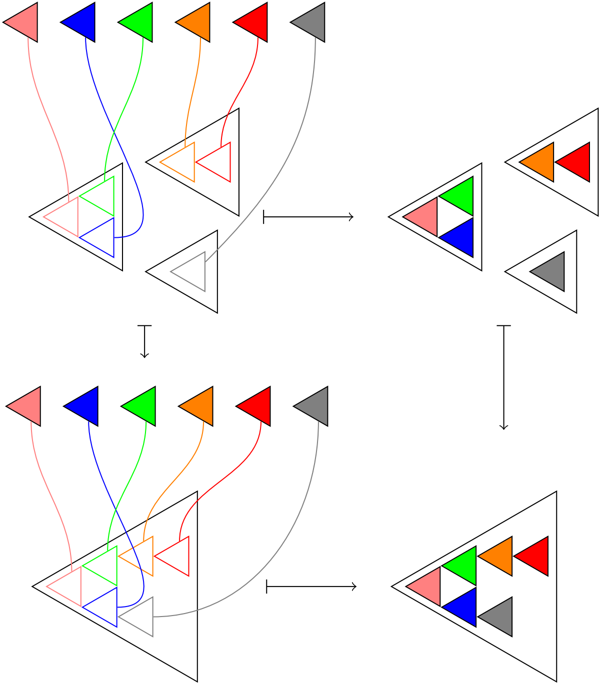
Theorem

There is a unique $\otimes$-monoid structure on $\mathcal{W} := \mathcal{F}_{\odot}(I_{\otimes})$, such that the unit of the adjunction $\eta : I_{\otimes} \to \mathcal{F}_{\odot}(I_{\otimes})$ is the unit of the multiplication $\nu : \mathcal{W} \otimes \mathcal{W} \to \mathcal{W}$, which in turn makes the following main diagram commute:

$$\begin{array}{ccc}
(\mathcal{W} \otimes \mathcal{W}) \odot (\mathcal{W} \otimes \mathcal{W}) & \xrightarrow{\quad \nu \odot \nu \quad} & \mathcal{W} \odot \mathcal{W} \\
\Big\downarrow{\scriptstyle \varphi} & & \Big\downarrow{\scriptstyle \mu} \\
(\mathcal{W} \odot \mathcal{W}) \otimes \mathcal{W} & & \\
\Big\downarrow{\scriptstyle \mu \otimes 1} & & \\
\mathcal{W} \otimes \mathcal{W} & \xrightarrow{\quad \nu \quad} & \mathcal{W}
\end{array}$$

μ - multiplication in free monoid

Main Diagram



Further Results

- Fibered version: $\mathcal{W}$ becomes functor of the base category.

Further Results

- Fibered version: $\mathcal{W}$ becomes functor of the base category.
- In $\mathbf{Sig}_{ma}$, multiplication in $\mathcal{W}(M)$ substitues formal composites of symbols for symbols. Essential feature of "function replacement".

Further Results

- Fibered version: $\mathcal{W}$ becomes functor of the base category.
- In $\mathbf{Sig}_{ma}$, multiplication in $\mathcal{W}(M)$ substitues formal composites of symbols for symbols. Essential feature of "function replacement".
- $\mathcal{W}(M)$ is naturally a monoid in $\mathbf{Sig}_a$ over M. Its algebras are equivalent to multicategories over M (with O fixed). Analogous to what Baez and Dolan want.

Further Results

- Fibered version: $\mathcal{W}$ becomes functor of the base category.
- In $\mathbf{Sig}_{ma}$, multiplication in $\mathcal{W}(M)$ substitues formal composites of symbols for symbols. Essential feature of "function replacement".
- $\mathcal{W}(M)$ is naturally a monoid in $\mathbf{Sig}_a$ over M. Its algebras are equivalent to multicategories over M (with O fixed). Analogous to what Baez and Dolan want.
- The amalgamation permutations of $\mathcal{W}(M)$ cannot be straightened out, in general, even if M is standard.

Opetopic Sets

an inductive definition

- X_0 is a set - the set of objects, or 0-cells.

Opetopic Sets

an inductive definition

- X_0 is a set - the set of objects, or 0-cells.
- S_0, the monoid of 0-pasting diagrams, is the pullback of the trivial monoid along $X_0 \to 1$:

$$\begin{array}{ccc} 1 & \longleftarrow & S_0 \\ \downarrow & & \downarrow \partial \\ \{*\}^\dagger & \xleftarrow{(!)^\dagger} & X_0^\dagger \end{array}$$

Opetopic Sets

an inductive definition

- Inductive step: X_n and S_n given

Opetopic Sets

an inductive definition

- Inductive step: X_n and S_n given
  1. X_{n+1} - chosen set of $(n+1)$-cells

Opetopic Sets

an inductive definition

- Inductive step: X_n and S_n given
  1. X_{n+1} - chosen set of $(n+1)$-cells
  2. $\vartheta_{n+1} : X_{n+1} \to S_n$ - give each cell domain and codomain.

Opetopic Sets

an inductive definition

- Inductive step: X_n and S_n given
  1. X_{n+1} - chosen set of $(n+1)$-cells
  2. $\vartheta_{n+1} : X_{n+1} \to S_n$ - give each cell domain and codomain.
  3. $\mathcal{W}(S_n)$ - calculate possible $(n+1)$-pasting diagrams.

Opetopic Sets

an inductive definition

- Inductive step: X_n and S_n given
  1. X_{n+1} - chosen set of $(n+1)$-cells
  2. $\vartheta_{n+1} : X_{n+1} \to S_n$ - give each cell domain and codomain.
  3. $\mathcal{W}(S_n)$ - calculate possible $(n+1)$-pasting diagrams.
  4. S_{n+1} is the pullback of $\mathcal{W}(S_n)$ along ϑ_{n+1}.

$$
\begin{array}{ccc}
\mathcal{W}(S_n) & \longleftarrow & S_{n+1} \\
\partial \downarrow & & \downarrow \partial \\
S_n^\dagger & \xleftarrow{(\vartheta_{n+1})^\dagger} & X_{n+1}^\dagger
\end{array}
$$

Opetopic Sets

an inductive definition

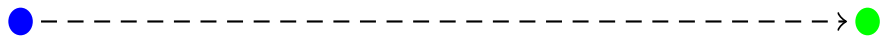
- Inductive step: X_n and S_n given
  1. X_{n+1} - chosen set of $(n+1)$-cells
  2. $\vartheta_{n+1} : X_{n+1} \to S_n$ - give each cell domain and codomain.
  3. $\mathcal{W}(S_n)$ - calculate possible $(n+1)$-pasting diagrams.
  4. S_{n+1} is the pullback of $\mathcal{W}(S_n)$ along ϑ_{n+1}.

$$\begin{array}{ccc} \mathcal{W}(S_n) & \longleftarrow & S_{n+1} \\ {\scriptstyle \partial}\downarrow & & \downarrow{\scriptstyle \partial} \\ S_n^{\dagger} & \xleftarrow{(\vartheta_{n+1})^{\dagger}} & X_{n+1}^{\dagger} \end{array}$$

This attaches cell names to the codomain and openings in the domain of every possible diagram.

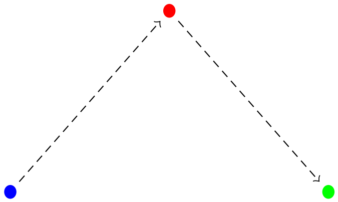
Opetopic Sets

in pictures



- An element of S_0

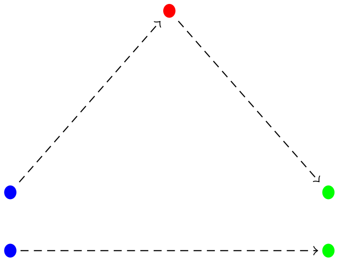
Opetopic Sets
in pictures



- Multiplication in S_0

Opetopic Sets

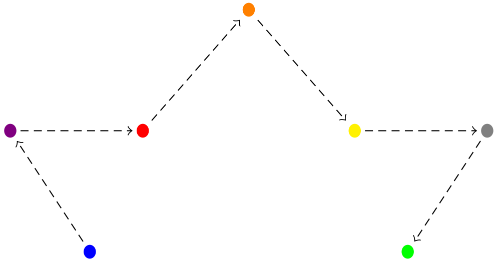
in pictures



- Multiplication in S_0

Opetopic Sets

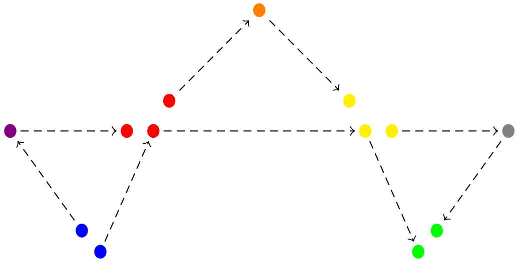
in pictures



- An element of $\mathcal{W}(S_0)$

Opetopic Sets

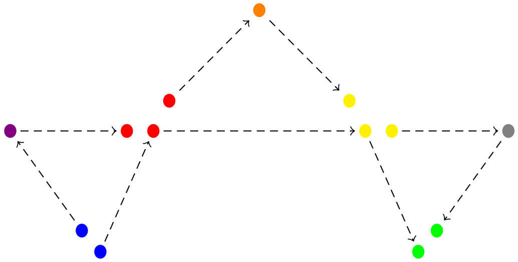
in pictures

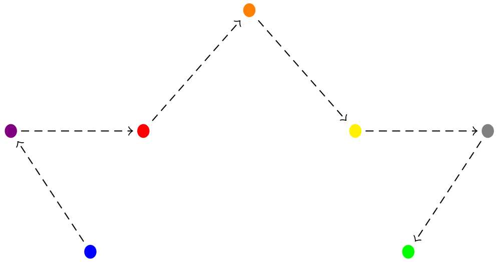


- Multiplication in $\mathcal{W}(S_0)$

Opetopic Sets

in pictures

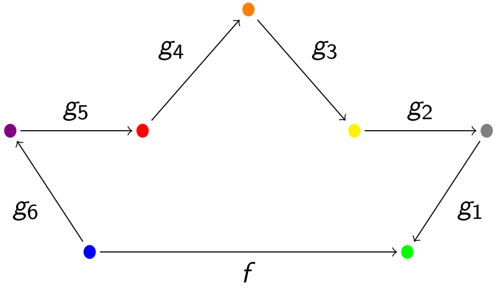




- Multiplication in $\mathcal{W}(S_0)$

Opetopic Sets

in pictures



- An element of S_1

Opetopic Sets

the category

A morphism of opetopic sets $X \to Y$:

Opetopic Sets

the category

A morphism of opetopic sets $X \to Y$:

- Maps of cells – functions $f_n : X_n \to Y_n$

Opetopic Sets

the category

A morphism of opetopic sets $X \to Y$:

- Maps of cells – functions $f_n : X_n \to Y_n$
- Compatible with forming pasting diagrams, taking domains and codomains.