Symmetry and Cauchy-completion

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Let $\mathcal{Q} = (\mathcal{Q}, \bigvee, \circ, 1)$ be a **quantale**, i.e. a poset which is a cocomplete monoidal closed category. More generally, $\mathcal{Q}$ can be a **quantaloid**.

A monotone function $\mathcal{Q} \longrightarrow \mathcal{Q}$: $f \mapsto f^{\circ}$ that satisfies $(gf)^{\circ} = f^{\circ} g^{\circ}$ and $f^{\circ\circ} = f$ is an **involution** on $\mathcal{Q}$.

Betti and Walters [1982] ask: "when is the Cauchy completion of a symmetric $\mathcal{Q}$-category again symmetric?"

Motivating examples:

- $([0, \infty], \bigwedge, +, 0)$ (cf. Lawvere's metric spaces),
- $\mathcal{R}(\mathcal{C}, J)$ (cf. Walters' sheaves on a site).

A $\mathcal{Q}$-category $\mathbb{C}$ is **Cauchy complete** if, for any $\mathcal{Q}$-category $\mathbb{X}$,

$$\mathsf{Cat}(\mathcal{Q})(\mathbb{X},\mathbb{C}) \longrightarrow \mathsf{Map}(\mathsf{Dist}(\mathcal{Q}))(\mathbb{X},\mathbb{C})\colon F \mapsto \mathbb{C}(-,F-)$$

is an equivalence.

There is an adjunction

$$\mathsf{Cat}_{\mathsf{cc}}(\mathcal{Q}) \underset{\text{full incl.}}{\overset{(-)_{\mathsf{cc}}}{\underset{\longrightarrow}{\overset{\longleftarrow}{\perp}}}} \mathsf{Cat}(\mathcal{Q})$$

Explicitly, $\mathbb{C}_{\mathsf{cc}}$ has

- objects: $\phi\colon \mathbb{1} \,{\circ}\!\!\!\longrightarrow \mathbb{C}$ with $\phi \dashv \phi^*$ ("Cauchy presheaves"),
- homs: $\mathbb{C}_{\mathsf{cc}}(\psi,\phi) = \psi^* \otimes \phi$.

The unit of the adjunction is $Y_{\mathbb{C}}\colon \mathbb{C} \longrightarrow \mathbb{C}_{\mathsf{cc}}\colon x \mapsto \mathbb{C}(-,x)$.

For an involutive $\mathcal{Q}$, a $\mathcal{Q}$-category $\mathbb{C}$ is **symmetric** if

$$\mathbb{C}(x,y) = \mathbb{C}(y,x)^{\circ}$$

for all $x, y \in \mathbb{C}$.

There is an adjunction

$$\mathsf{SymCat}(\mathcal{Q}) \underset{\text{full incl.}}{\overset{(-)_{\mathsf{s}}}{\underset{\longrightarrow}{\overset{\longleftarrow}{\top}}}} \mathsf{Cat}(\mathcal{Q})$$

Explicitly, $\mathbb{C}_{\mathsf{s}}$ has

- objects: the same as $\mathbb{C}$,
- homs: $\mathbb{C}_{\mathsf{s}}(x,y) = \mathbb{C}(x,y) \wedge \mathbb{C}(y,x)^{\circ}$.

The counit of the adjunction is $S_{\mathbb{C}} \colon \mathbb{C}_{\mathsf{s}} \longrightarrow \mathbb{C} \colon x \longrightarrow x$.

Betti and Walters [1982] ask about the **restriction**

$$\begin{array}{ccc} \mathsf{Cat}(\mathcal{Q}) & \xrightarrow{(-)_{\mathsf{cc}}} & \mathsf{Cat}(\mathcal{Q}) \\ {\scriptstyle \text{full incl.}}\uparrow & & \uparrow{\scriptstyle \text{full incl.}} \\ \mathsf{SymCat}(\mathcal{Q}) & \dashrightarrow & \mathsf{SymCat}(\mathcal{Q}) \end{array}$$

We give an elementary necessary-and-sufficient condition on $\mathcal{Q}$ to have an **extension**

$$\begin{array}{ccc} \mathsf{Cat}(\mathcal{Q}) & \xrightarrow{(-)_{\mathsf{cc}}} & \mathsf{Cat}(\mathcal{Q}) \\ {\scriptstyle (-)_{\mathsf{s}}}\downarrow & & \downarrow{\scriptstyle (-)_{\mathsf{s}}} \\ \mathsf{SymCat}(\mathcal{Q}) & \dashrightarrow & \mathsf{SymCat}(\mathcal{Q}) \end{array}$$

which implies the restriction.

For an involutive $\mathcal{Q}$, also $\mathsf{SymDist}(\mathcal{Q})$ is involutive:

$$(\Phi\colon \mathbb{A} \mathrel{-\!\!\circ\!\!\!\rightarrow} \mathbb{B})^{\circ} = \Phi^{\circ}\colon \mathbb{B} \mathrel{-\!\!\circ\!\!\!\rightarrow} \mathbb{A} \text{ with } \Phi^{\circ}(x,y) = \Phi(y,x)^{\circ}.$$

A distributor $\Phi\colon \mathbb{A} \mathrel{-\!\!\circ\!\!\!\rightarrow} \mathbb{B}$ between symmetric $\mathcal{Q}$-categories is a **symmetric left adjoint** if $\Phi \dashv \Phi^{\circ}$.

We then have

$$\begin{array}{ccc}
\mathsf{Cat}(\mathcal{Q}) & \longrightarrow & \mathsf{Map}(\mathsf{Dist}(\mathcal{Q})) \\
\text{full incl.} \uparrow & & \uparrow \text{(non-full) incl.} \\
\mathsf{SymCat}(\mathcal{Q}) & \longrightarrow & \mathsf{SymMap}(\mathsf{SymDist}(\mathcal{Q}))
\end{array}$$

A symmetric $\mathcal{Q}$-category $\mathbb{A}$ is **symetrically complete** if, for any symmetric $\mathcal{Q}$-category $\mathbb{X}$,

$$\mathsf{SymCat}(\mathcal{Q})(\mathbb{X},\mathbb{A}) \longrightarrow \mathsf{SymMap}(\mathsf{SymDist}(\mathcal{Q}))(\mathbb{X},\mathbb{A})\colon F \mapsto \mathbb{A}(-,F-)$$

is an equivalence.

There is an adjunction

$$\mathsf{SymCat}_{\mathsf{sc}}(\mathcal{Q}) \underset{\text{full incl.}}{\overset{(-)_{\mathsf{sc}}}{\underset{\longrightarrow}{\overset{\longleftarrow}{\perp}}}} \mathsf{SymCat}(\mathcal{Q})$$

Explicitly, $\mathbb{A}_{\mathsf{sc}}$ has

- objects: $\phi\colon \mathbb{1} \multimap \mathbb{A}$ with $\phi \dashv \phi^{\mathsf{o}}$ ("symmetric Cauchy presheaves"),
- homs: $\mathbb{A}_{\mathsf{sc}}(\psi,\phi) = \psi^{\mathsf{o}} \otimes \phi$.

The unit of the adjunction is $Y_{\mathbb{A}}\colon \mathbb{A} \longrightarrow \mathbb{A}_{\mathsf{sc}}\colon x \mapsto \mathbb{A}(-,x)$.

By construction we have

$$\begin{array}{ccc} \mathsf{Cat}(\mathcal{Q}) & \xrightarrow{(-)_{\mathsf{cc}}} & \mathsf{Cat}(\mathcal{Q}) \\ {\scriptstyle \text{full incl.}}\uparrow & \overset{K}{\Nwarrow} & \uparrow{\scriptstyle \text{full incl.}} \\ \mathsf{SymCat}(\mathcal{Q}) & \xrightarrow[(-)_{\mathsf{sc}}]{} & \mathsf{SymCat}(\mathcal{Q}) \end{array}$$

with components $K_{\mathbb{A}} \colon \mathbb{A}_{\mathsf{sc}} \longrightarrow \mathbb{A}_{\mathsf{cc}} \colon \phi \mapsto \phi$.

Its mate is

$$\begin{array}{ccc} \mathsf{Cat}(\mathcal{Q}) & \xrightarrow{(-)_{\mathsf{cc}}} & \mathsf{Cat}(\mathcal{Q}) \\ {\scriptstyle (-)_{\mathsf{s}}}\downarrow & \overset{L}{\Nearrow} & \downarrow{\scriptstyle (-)_{\mathsf{s}}} \\ \mathsf{SymCat}(\mathcal{Q}) & \xrightarrow[(-)_{\mathsf{sc}}]{} & \mathsf{SymCat}(\mathcal{Q}) \end{array}$$

with components $L_{\mathbb{C}} \colon (\mathbb{C}_{\mathsf{s}})_{\mathsf{sc}} \longrightarrow (\mathbb{C}_{\mathsf{cc}})_{\mathsf{s}} \colon \phi \mapsto \mathbb{C}(-, S_{\mathbb{C}}-) \otimes \phi$.

Theorem: the following are equivalent:

1. isomorphism:

$$\begin{array}{ccc}
\mathsf{Cat}(\mathcal{Q}) & \xrightarrow{(-)_{\mathsf{cc}}} & \mathsf{Cat}(\mathcal{Q}) \\
{\scriptstyle (-)_{\mathsf{s}}}\downarrow & \overset{L}{\nearrow\!\!\!\nearrow}\,{\scriptstyle \cong} & \downarrow{\scriptstyle (-)_{\mathsf{s}}} \\
\mathsf{SymCat}(\mathcal{Q}) & \xrightarrow[(-)_{\mathsf{sc}}]{} & \mathsf{SymCat}(\mathcal{Q})
\end{array}$$

2. adjoint:

$$\begin{array}{ccc}
\mathsf{Cat}(\mathcal{Q}) & \longrightarrow & \mathsf{Map}(\mathsf{Dist}(\mathcal{Q})) \\
{\scriptstyle \mathsf{incl.}}\uparrow \dashv \downarrow{\scriptstyle (-)_{\mathsf{s}}} & & {\scriptstyle \mathsf{incl.}}\uparrow \dashv \downarrow{\scriptstyle (-)_{\mathsf{s}}} \\
\mathsf{SymCat}(\mathcal{Q}) & \longrightarrow & \mathsf{SymMap}(\mathsf{SymDist}(\mathcal{Q}))
\end{array}$$

3. for each $(f_i, g_i)_{i \in I}$ in $\mathcal{Q}$,

$$\left.\begin{array}{l}
f_k \circ g_j \circ f_j \leq f_k \\
g_j \circ f_j \circ g_k \leq g_k \\
1 \leq \bigvee_i g_i \circ f_i
\end{array}\right\} \implies 1 \leq \bigvee_i (g_i \wedge f_i^{\circ}) \circ (g_i^{\circ} \wedge f_i)$$

The second part of the Theorem speaks of the **symmetrisation** of left adjoint distributors.

More precisely, given $\Phi\colon \mathbb{C} \multimap \mathbb{D}$ in $\mathsf{Map}(\mathsf{Dist}(\mathcal{Q}))$, define $\Phi_{\mathsf{s}}\colon \mathbb{C}_{\mathsf{s}} \multimap \mathbb{D}_{\mathsf{s}}$ in $\mathsf{SymDist}(\mathcal{Q})$ as

$$\Phi_{\mathsf{s}} := \Big(\mathbb{D}(S_{\mathbb{D}}-,-)\otimes\Phi\otimes\mathbb{C}(-,S_{\mathbb{C}}-)\Big)\wedge\Big(\mathbb{C}(S_{\mathbb{C}}-,-)\otimes\Phi^*\otimes\mathbb{D}(-,S_{\mathbb{D}}-)\Big)^{\circ}$$

The statements in the Theorem are all equivalent to:

4. If $\mathbb{C} \underset{\Phi^*}{\overset{\Phi}{\underset{\perp}{\rightleftarrows}}} \mathbb{D}$ then $\mathbb{C}_{\mathsf{s}} \underset{(\Phi_{\mathsf{s}})^{\circ}}{\overset{\Phi_{\mathsf{s}}}{\underset{\perp}{\rightleftarrows}}} \mathbb{D}_{\mathsf{s}}$

Corollary: in this case, the following diagrams commute:

$$\begin{CD}
\mathsf{Cat}(\mathcal{Q}) @>{(-)_{\mathsf{cc}}}>> \mathsf{Cat}(\mathcal{Q}) \\
@A{\text{incl.}}AA @AA{\text{incl.}}A \\
\mathsf{SymCat}(\mathcal{Q}) @>>{(-)_{\mathsf{cc}}}> \mathsf{SymCat}(\mathcal{Q})
\end{CD}
\qquad
\begin{CD}
\mathsf{Cat}(\mathcal{Q}) @>{(-)_{\mathsf{s}}}>> \mathsf{Cat}(\mathcal{Q}) \\
@A{\text{incl.}}AA @AA{\text{incl.}}A \\
\mathsf{Cat}_{\mathsf{cc}}(\mathcal{Q}) @>>{(-)_{\mathsf{s}}}> \mathsf{Cat}_{\mathsf{cc}}(\mathcal{Q})
\end{CD}$$

That is to say, there is a **distributive law** of the monad $(-)_{\mathsf{cc}}$ over the comonad $(-)_{\mathsf{s}}$.

Examples:

(1) Lawvere's quantale of positive reals $([0, \infty], \bigwedge, +0)$

(2) Free quantaloid on a groupoid, with "canonical involution"

(3) Walters' quantaloid of relations $\mathcal{R}(\mathcal{C}, J)$

(4) Any locally localic, modular quantaloid

(5) Betti and Walters' counterexample: free quantale on commutative group $\{1, a, b\}$ with $aa = b$, $ab = 1$ and $bb = a$, with identity involution

To do:

(6) Find elementary necessary-and-sufficient condition for distributive law.

(7) Generalise to general cocompletion KZ-doctrines instead of $(-)_{\mathsf{cc}}$.