The algebra of complicial sets

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Complicial sets and ω-categories

- A *complicial set* is a simplicial set together with a distinguished class of *thin* elements of positive dimension,
- such that certain *complicial* horns have unique thin fillers
- and such that a few other simple conditions are satisfied.
- Theorem (Verity). *Complicial sets are equivalent to strict* ω*-categories*.

Aims of this talk

- Describe complicial sets algebraically by operations and identities.
- Describe complicial sets as functors.

Thin fillers as unary and binary operations

- Let x be the thin filler of a complicial horn opposite vertex i.
- For $j<i-1$ and for $j>i+1$ the face $\partial_j x$ is itself the thin filler of a complicial horn, etc.
- If $i=0$ then x is determined by $\partial_1 x$
- and in fact $x=\varepsilon_0\partial_1 x$.
- If $i=\dim x$ then $x=\varepsilon_{i-1}\partial_{i-1}x$.
- If $0<i<\dim x$ then x is determined by $\partial_{i-1}x$ and $\partial_{i+1}x$, say $x = \partial_{i+1}x \wedge_{i-1} \partial_{i-1}x$.

Algebraic description 1

- A complicial set X is a sequence of sets $X_0, X_1, \ldots$ together with
- *faces* $\partial_i x \in X_{m-1}$ ($x \in X_m$, $m>0$, $0 \le i \le m$),
- *degeneracies* $\varepsilon_i x \in X_{m+1}$ ($x \in X_m$, $0 \le i \le m$),
- *wedges* $x \wedge_i y \in X_{m+1}$ ($x, y \in X_m$, $0 \le i \le m-1$, $\partial_i x = \partial_{i+1} y$),
- such that …

Algebraic description 2

- $\partial_i \partial_j x = \partial_{j-1} \partial_i x$ ($i<j$),
- $\partial_i \varepsilon_i x = \partial_{i+1} \varepsilon_i x = x$,
- $\partial_{i+2}(x \wedge_i y) = x$, $\partial_i(x \wedge_i y) = y$,
- and …

Algebraic description 3

- $\varepsilon_i \partial_{i+1} x \wedge_i x = \varepsilon_i x$,
- $x \wedge_i \varepsilon_i \partial_i x = \varepsilon_{i+1} x$,
- $\partial_j (x \wedge_i y) = \partial_j x \wedge_{i-1} \partial_j y$ ($j<i$),
- $\partial_j (x \wedge_i y) = \partial_{j-1} x \wedge_i \partial_{j-1} y$ ($j>i+2$),
- if $A = b \wedge_i (y \wedge_i z)$ then $A = (\partial_{i+2} b \wedge_i y) \wedge_{i+1} \partial_{i+1} A$,
- if $A = (x \wedge_i y) \wedge_{i+1} c$ then $A = \partial_{i+2} A \wedge_i (y \wedge_{i+1} \partial_i c)$,
- $[x \wedge_i \partial_{i+1}(y \wedge_i z)] \wedge_i (y \wedge_i z) = (x \wedge_i y) \wedge_{i+1} [\partial_{i+1}(x \wedge_i y) \wedge_i z]$,
- if $B = \partial_{i+2}[(x \wedge_{i+1} y) \wedge_{i+1} (y \wedge_i z)]$ then
  $$B \wedge_i (w \wedge_{i+1} \partial_i B) = (\partial_{i+3} B \wedge_i w) \wedge_{i+2} B,$$
- $(x \wedge_i y) \wedge_j (z \wedge_i w) = (x \wedge_{j-1} z) \wedge_i (y \wedge_{j-1} w)$ ($i \le j-3$).

Functorial description

- The *category of orientals* $\mathcal{O}$ has objects $0,1,\ldots$ and morphism sets $\mathcal{O}(m,n)$,
- where a morphism in $\mathcal{O}(m,n)$ is an augmentation-preserving chain map from the chain complex of the m-simplex to the chain complex of the n-simplex taking basis elements to sums of basis elements.
- Let an *oriental set* be a contravariant functor from $\mathcal{O}$ to sets.
- Then a complicial set is an oriental set satisfying certain limit conditions.
- A strict ω-category determines an oriental set (in fact a complicial set), because the category $\mathcal{O}$ is a full subcategory of the category of strict ω-categories.

Operations in oriental sets

- An *oriental set* $X=(X_0,X_1,\ldots)$ has face and degeneracy operations
- and also idempotent *angle* operations $\lambda_i:X_{m+1}\rightarrow X_{m+1}$ $(0\leq i\leq m-1)$
- given by projection onto the union of the faces opposite i and $i+2$.

Limit conditions 1

- An oriental set X is a complicial set if it has the following properties.
- For $0 \leq i \leq m-1$, if $x, y \in X_m$ and $\partial_i x = \partial_{i+1} y$, then there is a unique element $x \wedge_i y \in X_{m+1}$ such that

$$\lambda_i(x \wedge_i y) = x \wedge_i y,\ \partial_{i+2}(x \wedge_i y) = x,\ \partial_i(x \wedge_i y) = y.$$

- ...

Limit conditions 2

- For $0 \leq i \leq m-2$, if b, $y \wedge_i z$ ε X_m and $\partial_i b = \partial_{i+1}(y \wedge_i z)$, then there is a unique A ε X_{m+1} such that

$$\lambda_i A = \lambda_{i+1} A = A,\ \partial_{i+2} A = b,\ \partial_i A = (y \wedge_i z).$$

- For $0 \leq i \leq m-2$, if $x \wedge_i y$, c ε X_m and $\partial_{i+1}(x \wedge_i y) = \partial_{i+2} c$, then there is a unique A ε X_{m+1} such that

$$\lambda_i A = \lambda_{i+1} A = A,\ \partial_{i+3} A = x \wedge_i y,\ \partial_{i+1} A = c.$$

- For $0 \leq i \leq m-2$, if $B = \partial_{i+2}[(x \wedge_{i+1} y) \wedge_{i+1} (y \wedge_i z)]$ ε X_{m+1} and w ε X_m and $\partial_i \partial_{i+3} B = \partial_{i+1} w$, then there is a unique A ε X_{m+2} such that

$$\lambda_i A = \lambda_{i+2} A = A,\ \partial_{i+2} A = B,\ \partial_i \partial_{i+4} A = w.$$

Part of proof

- The universal oriental set $\mathcal{O}(-,n)$ is the complicial set in the algebraic sense freely generated by the identity morphism ι_n in $\mathcal{O}(n,n)$.
- In particular, every morphism in $\mathcal{O}(-,n)$ can be expressed as a word in ι_n.
- For a morphism in in $\mathcal{O}$, let the *rank* be the number of vertices which are not sent to the last vertex,
- and let the *corank* be the number of vertices which are sent to the last vertex.
- Let x be a morphism of rank r and corank s in $\mathcal{O}(-,n)$.
- We get a canonical expression for x as a word in ι_n by induction on n, and for fixed n by induction on s.

Canonical words

- If $r=0$ then $x=\varepsilon_0^{s-1}\partial_0^{n}\iota_n$.
- If $s=0$ then $x=\partial_r\gamma\pi x$, where $\pi x \in \mathcal{O}(-,n-1)$ and γ is a cone operation taking words in ι_{n-1} to the corresponding words in ι_n.
- If $r>0$ and $s>0$ then

$$x=\partial_r(\alpha_{r-1}x \wedge_{r-1}{}^{1})\partial_{r-1}(\alpha_{r-2}x \wedge_{r-2}{}^{2})\ldots\partial_1(\alpha_0 x \wedge_0{}^{r})\varepsilon_r{}^{s-1}\gamma\pi x,$$

- where $\gamma\pi x$ is as before,
- $\alpha_p x$ is of corank $s-1$ in $\mathcal{O}(-,n)$,
- and $(u \wedge_k{}^{l})w$ denotes

$$u(\wedge_k \partial_{k+1}{}^{l-1}w)\ldots(\wedge_k \partial_{k+1}w)(\wedge_k w).$$