Autonomous categories with self-dual objects

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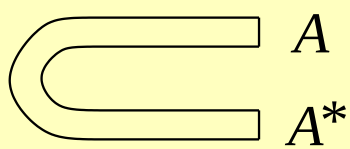
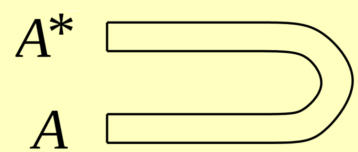
Autonomous categories

A *right dual* for an object A in a monoidal category is given by $(A^*, \eta_A, \epsilon_A)$, where $\eta_A : \mathrm{I} \to A^* \otimes A$ and $\epsilon_A : A \otimes A^* \to \mathrm{I}$ and

$$\begin{array}{ccc} A & \xrightarrow{A\otimes\eta_A} & A \otimes A^* \otimes A \\ & \searrow^{\mathrm{id}} & \downarrow{\scriptstyle \epsilon_A \otimes A} \\ & & A, \end{array} \qquad \begin{array}{ccc} A^* & \xrightarrow{\eta_A\otimes A^*} & A^* \otimes A \otimes A^* \\ & \searrow^{\mathrm{id}} & \downarrow{\scriptstyle A^* \otimes \epsilon_A} \\ & & A^*. \end{array}$$

A monoidal category is *right autonomous* if every object A has a right dual.

Dual $\quad \eta_A =$ 

, $\quad \epsilon_A =$ 

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Autonomous = *right and left autonomous*.

Remark: no coherence required!

- The definition of autonomous category requires *no further coherence conditions*. In fact, it is a *property* of monoidal categories: the triple $(A^*, \eta_A, \epsilon_A)$, if it exists, is uniquely determined by A up to isomorphism.

- Why no coherence conditions? E.g. the dual of $A \otimes B$ is related to the duals of A and B by

$$k_{A,B} \; : \; (A \otimes B)^* \cong B^* \otimes A^*.$$

  There is a potential coherence axiom:

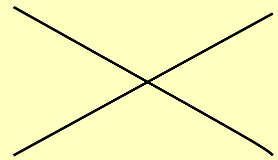
$$\begin{array}{ccc}
I & \xrightarrow{\eta_{A \otimes B}} & A \otimes B \otimes (A \otimes B)^* \\
\downarrow{\scriptstyle \eta_A} & & \downarrow{\scriptstyle A \otimes B \otimes k_{A,B}} \\
A \otimes A^* & \xrightarrow{A \otimes \eta_B \otimes A^*} & A \otimes B \otimes B^* \otimes A^*.
\end{array}$$

  However, this is simply the *definition* of $k_{A,B}$.

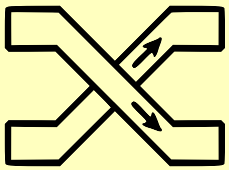
- Similarly $k_I : I^* \cong I$.

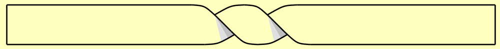
Compact closed and tortile categories.

- Compact closed = autonomous and symmetric (no additional axioms)

Symmetry $c_{A,B} =$ 

- Tortile = autonomous and balanced (only one additional axiom: $\theta_A^* = \theta_{A^*}$)

Braiding $c_{A,B} =$ 

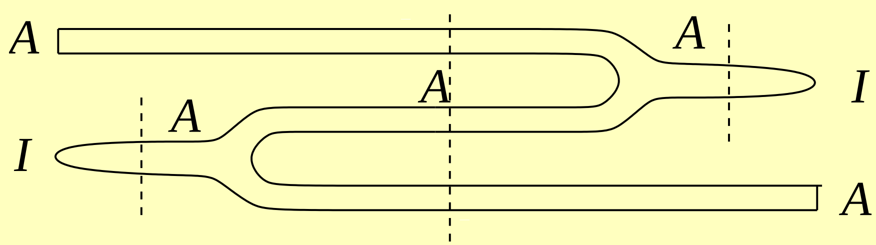
Twist $\theta_A =$ 

Self-dual objects

Recently, there has been some interest in objects in autonomous categories being self-dual:

$$A \cong A^*$$

- Certain irreducible group representations are self-dual iff they are real or quaternionic — but not complex.
- In quantum computing: identify a qubit with a co-qubit.
- Involutive categories [Egger 2007–2010; Jacobs 2010]
- Frobenius algebra structures induce self-dual autonomous structure [e.g. Coecke-Pavlovic; see Carboni-Walters-Wood]:



Extent of duality

- Sometimes, one is interested in an *individual* object that is self-dual (e.g., $\mathbb{C}^2$ in FdHilb).

- Sometimes, one is interested in a category where *all* objects are canonically self-dual (e.g., finite dimensional real inner product spaces).

- Other times, one is interested in situations where the objects are not *canonically* self-dual, but can be equipped with a *chosen* self-duality (e.g., finite dimensional complex Hilbert spaces with chosen bases — or chosen classical structures).

The naïve approach

The definition of autonomous categories was so simple.

$$\eta : I \to A^* \otimes A \qquad \epsilon : A \otimes A^* \to I$$

$$\begin{array}{ccc} A & \xrightarrow{A \otimes \eta} & A \otimes A^* \otimes A \\ & \searrow^{\mathrm{id}} & \downarrow{\scriptstyle \epsilon \otimes A} \\ & & A, \end{array} \qquad \begin{array}{ccc} A^* & \xrightarrow{\eta \otimes A^*} & A^* \otimes A \otimes A^* \\ & \searrow^{\mathrm{id}} & \downarrow{\scriptstyle A^* \otimes \epsilon} \\ & & A^*. \end{array}$$

We could just postulate $A = A^*$:

$$\hat{\eta} : I \to A \otimes A \qquad \hat{\epsilon} : A \otimes A \to I$$

$$\begin{array}{ccc} A & \xrightarrow{A \otimes \hat{\eta}} & A \otimes A \otimes A \\ & \searrow^{\mathrm{id}} & \downarrow{\scriptstyle \hat{\epsilon} \otimes A} \\ & & A, \end{array} \qquad \begin{array}{ccc} A & \xrightarrow{\hat{\eta} \otimes A} & A \otimes A \otimes A \\ & \searrow^{\mathrm{id}} & \downarrow{\scriptstyle A \otimes \hat{\epsilon}} \\ & & A. \end{array}$$

Coherence?

A nice feature of the definition of autonomous category was that is required no coherence condition. However, if we postulate $A = A^*$, this is no longer the case!

For example, in any autonomous category, we have

$$k_{A,B} \; : \; (A \otimes B)^* \cong B^* \otimes A^*.$$

If we require $A^* = A$ for all objects, we get:

$$k_{A,B} \; : \; A \otimes B \cong B \otimes A.$$

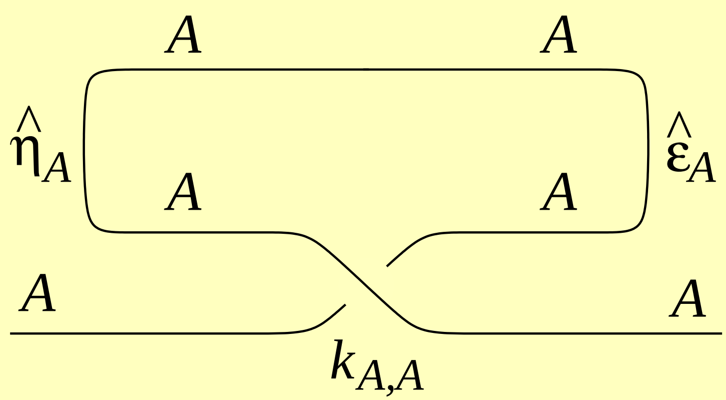
In all likelihood, we would like $k_{A,B}$ to satisfy additional properties, for example, we would like it to be a *braiding*.

Coherence!

We therefore get a *braiding*

$$k_{A,B} \ : \ A \otimes B \cong B \otimes A.$$

Moreover, from this diagram:



we get a *twist* $\theta_A : A \to A$. We probably want this to satisfy the axioms of a balanced monoidal category.

Moreover, we would like the axioms to be such that we can use a *graphical language* to reason about them.

These axioms is what this talk is about.

The half-twist map

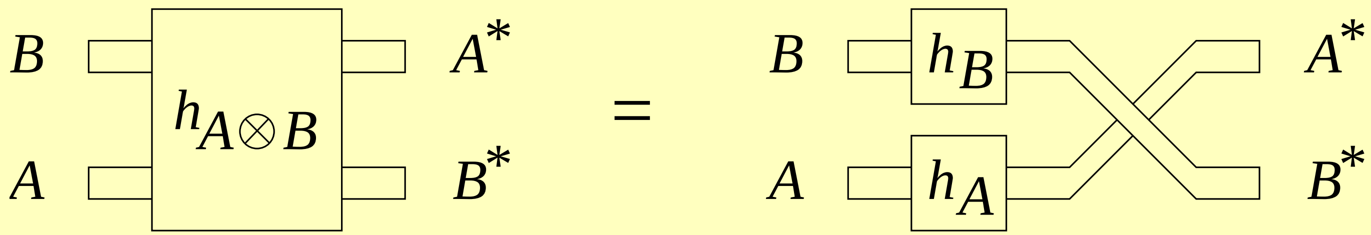
Rather than asking for equality $A = A^*$, our initial approach is to ask for an isomorphism

$$h_A : A \cong A^*,$$

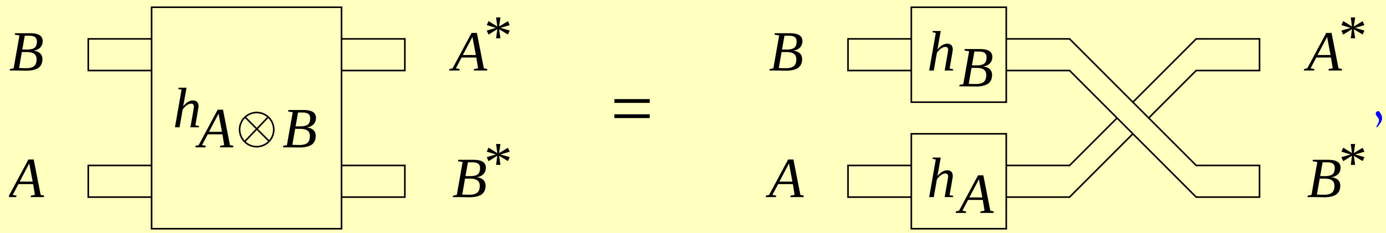
and axiomatizing its desirable properties. We will get back to strict equality later.

How should $h_A : A \to A^*$ look in the graphical language?

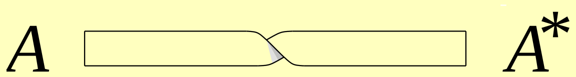
- Initial impulse: write h_A like an identity.
- However, consider $h_{A\otimes B} : A \otimes B \to (A \otimes B)^* \cong B^* \otimes A^*$. This can't be equal to $h_A \otimes h_B : A \otimes B \to A^* \otimes B^*$. Rather, we expect:



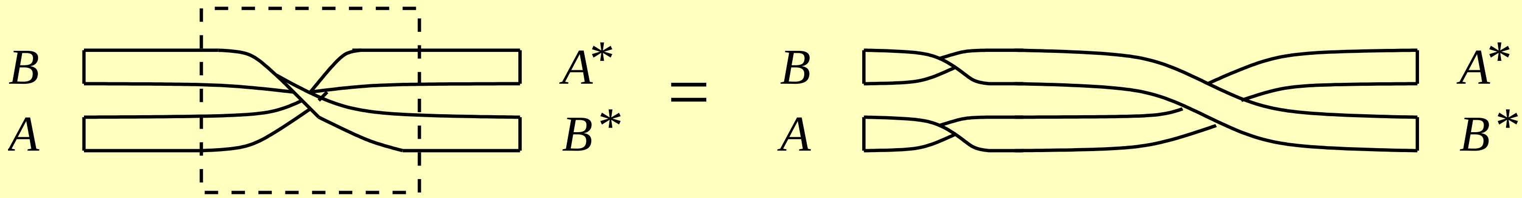
To satisfy



it is reasonable to represent h as a *half-twist*:



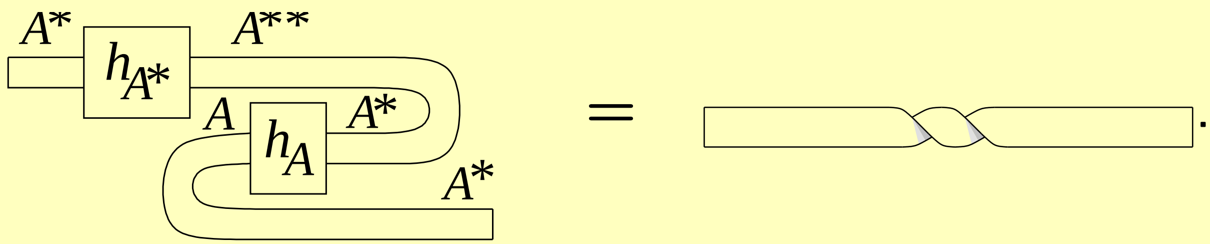
The above equation then becomes:

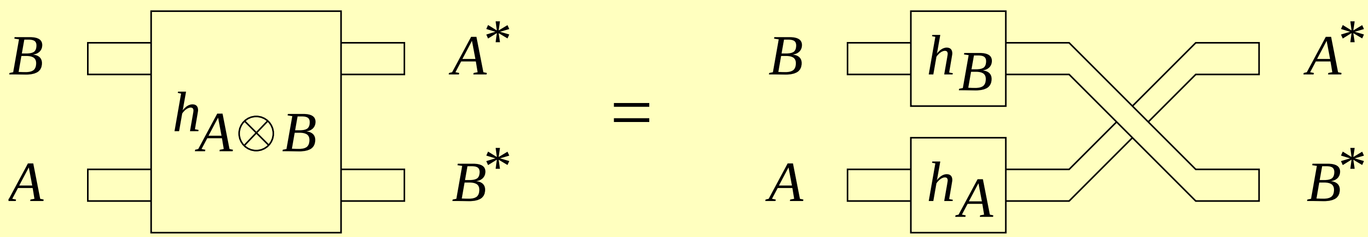


Guided by the graphical language (framed tangles up to ambient framed isotopy), we formulate sound axioms.

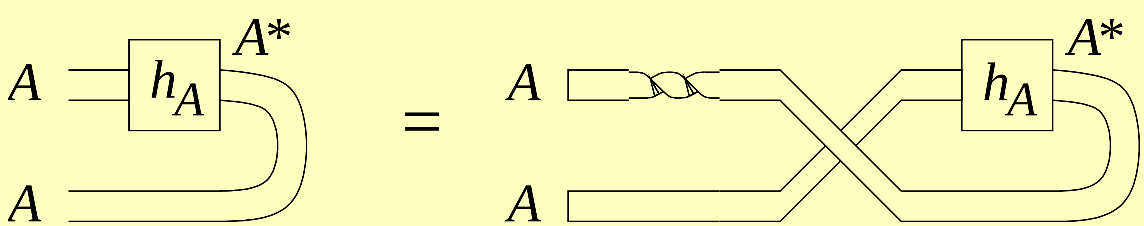
Definition. Start with a tortile category. A *self-duality structure* is given by a family of morphisms $h_A : A \to A^*$ and 5 axioms:

(T1) h_A is an isomorphism.

(T2) 

(T3) 

(T4) $h_I = k_I : I \xrightarrow{\cong} I^*$.

(T5) 

Remarks.

- The axioms are sound for the graphical language.
- Axioms (T2)–(T4) are classic examples of coherence axioms: in fact, they can be taken as the *definition* of h_{A^*}, $h_{A\otimes B}$, and h_I, when the objects are generated freely.
- If (T5) holds for h_A and h_B, then also for h_{A^*}, $h_{A\otimes B}$, and h_I.
- (T1) is a consequence of (T2) and (T5). The remaining axioms are independent. (Proof by counterexamples).

Theorem (coherence). The axioms are also *complete* for the graphical language. Framed tangles up to ambient framed isotopy form the *free* such category.

Natural examples:

- finite dimensional real inner product spaces
- finite sets and relations

Non-natural examples:

- finite dimensional Hilbert spaces with chosen bases or classical structures.
- From any tortile category $\mathbf{C}$, construct another one $\mathbf{D}$ with self-duality: objects are pairs (A, h) where $\mathrm{h} : A \to A^*$ satisfies (T5).

Alternative axioms (based on right autonomous category)

If a tortile structure is not given *a priori*, we can still postulate $h_A : A \to A^*$ and add axioms so that the *a posteriori* induced structure is tortile.

Definition. Start with a right autonomous category. A *self-duality structure* is given by a family $h_A : A \to A^*$ and 7 axioms. We write

$$f_\sharp = A^* \xrightarrow{h_A^{-1}} A \xrightarrow{f} B \xrightarrow{h_B} B^*.$$

(A1) h_A is an isomorphism.

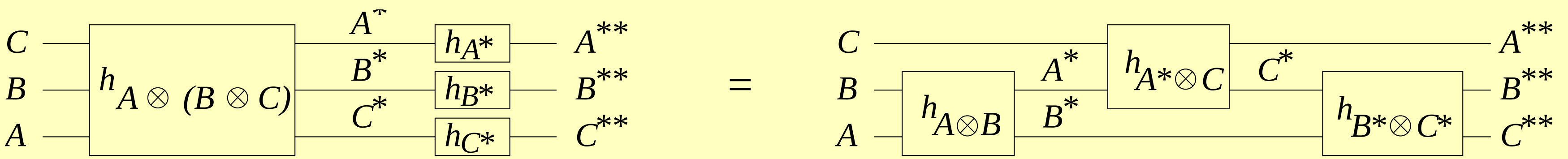
(A2) $h_I = k_I : I \xrightarrow{\cong} I^*$

(A3) $(f^*)_\sharp = (f_\sharp)^*$.

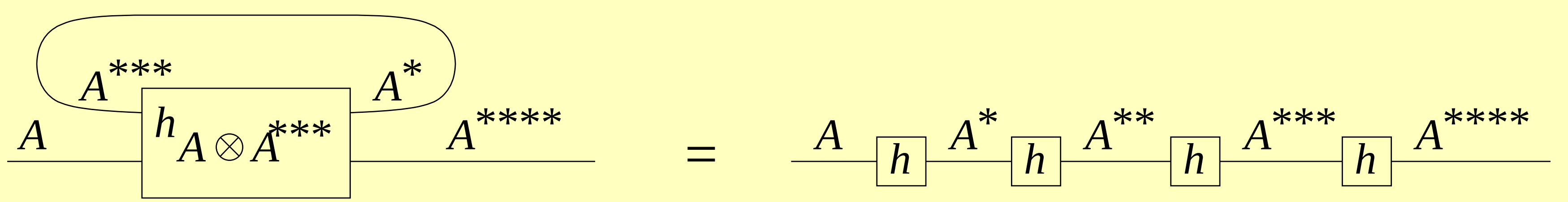
(A4) $$\begin{array}{ccc} (A \otimes B)^* & \xrightarrow{(f \otimes g)_\sharp} & (A' \otimes B')^* \\ \downarrow k_{A,B} & & \downarrow k_{A',B'} \\ B^* \otimes A^* & \xrightarrow{g_\sharp \otimes f_\sharp} & B'^* \otimes A'^*. \end{array}$$

(A5) $\alpha^* = (\alpha_\sharp)^{-1}$, where $\alpha : (A \otimes B) \otimes C \to A \otimes (B \otimes C)$.

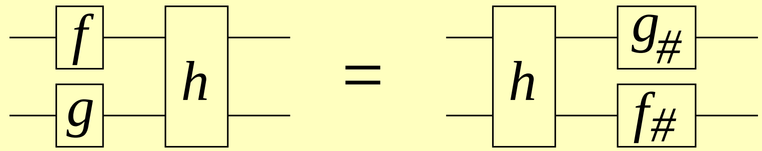
(A6)



(A7)



Some consequences



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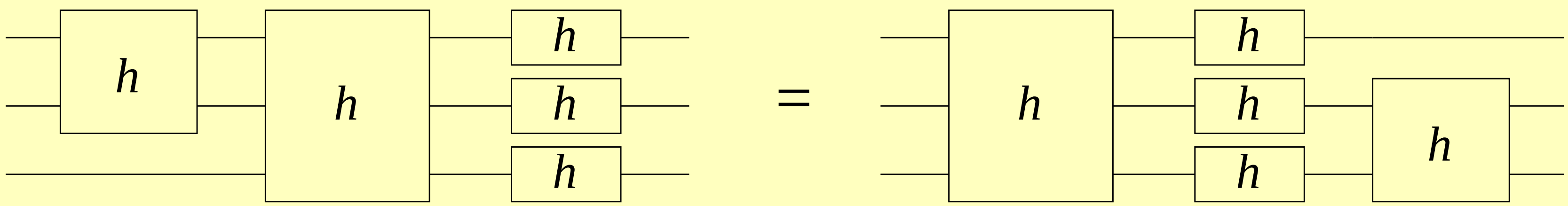

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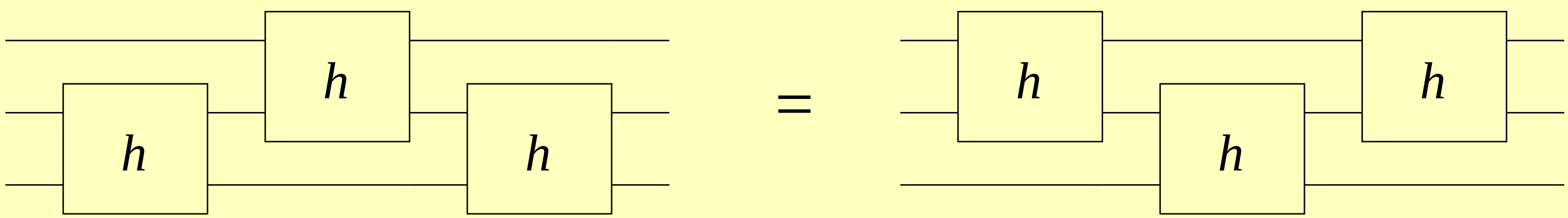

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Theorem. There is a one-to-one correspondence between:

- tortile categories with a self-dual structure satisfying (T1)–(T5), and
- right autonomous categories with a self-dual structure satisfying (A1)–(A8).

Remark.

There is a *dagger structure* induced by

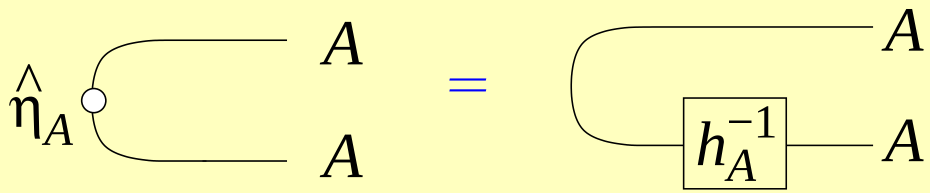
$$f^\dagger = B \xrightarrow{h_B} B^* \xrightarrow{f^*} A^* \xrightarrow{h_A^{-1}} A.$$

HOWEVER: It is not a "good" dagger structure. *The braiding and twist maps fail to be unitary*. In particular, it is not a dagger tortile category.

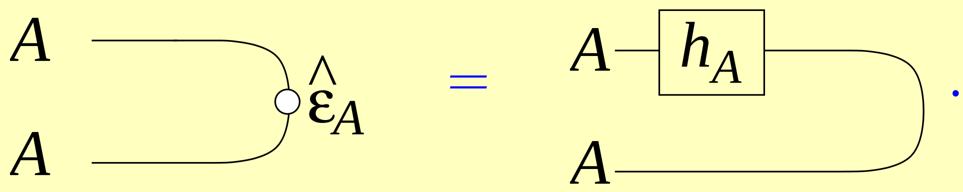
Also, on the category of finite dimensional Hilbert spaces (with chosen bases or classical structures), the induced dagger does **not** coincide with the "natural" dagger defined by adjoints. It is the identity (rather than complex conjugation) on scalars.

Strict self-duality

Given a self-dual structure on a right autonomous category, we can define



and



- Any two of $\hat{\eta}_A$, η_A, and h_A determine the third uniquely.
- Therefore, we can give an equivalent axiomatization in terms of $\hat{\eta}_A$, $\hat{\epsilon}_A$, and h_A.

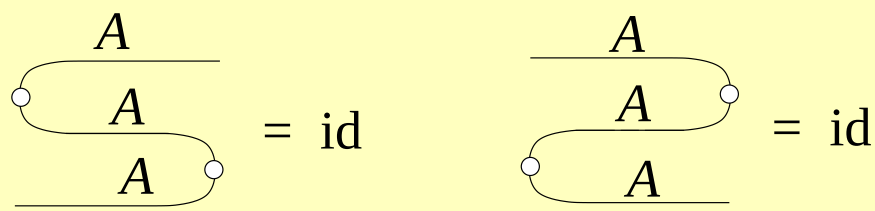
Alternative axioms 2 (based on monoidal category)

Definition. Start with a monoidal category. A *self-duality structure* is given an object operation A^* and by families of morphisms

$$\hat{\eta} : I \to A \otimes A \qquad \hat{\epsilon} : A \otimes A \to I \qquad h_A : A \to A^*$$

subject to 10 axioms:

(M1) (autonomous)

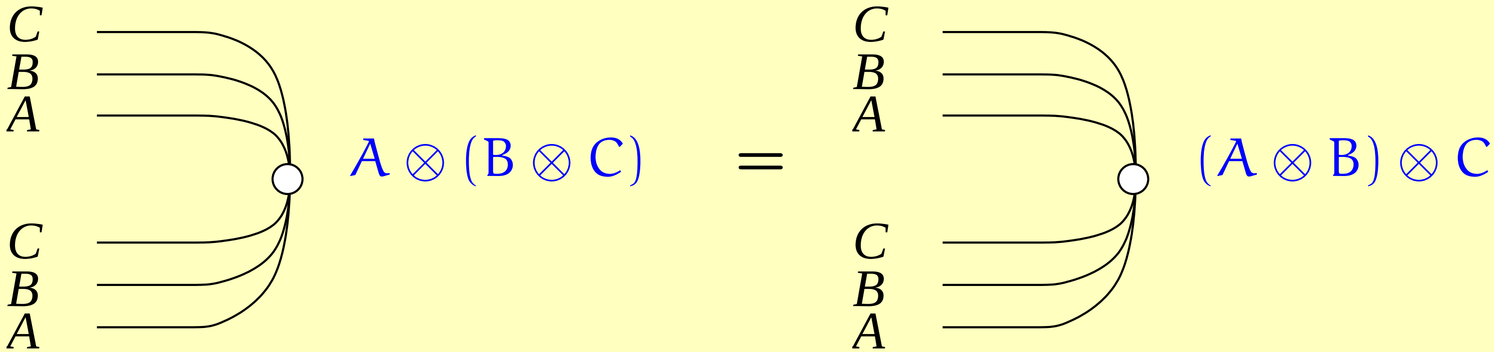


(M2) $\hat{\eta}_I = I \xrightarrow{\cong} I \otimes I$.

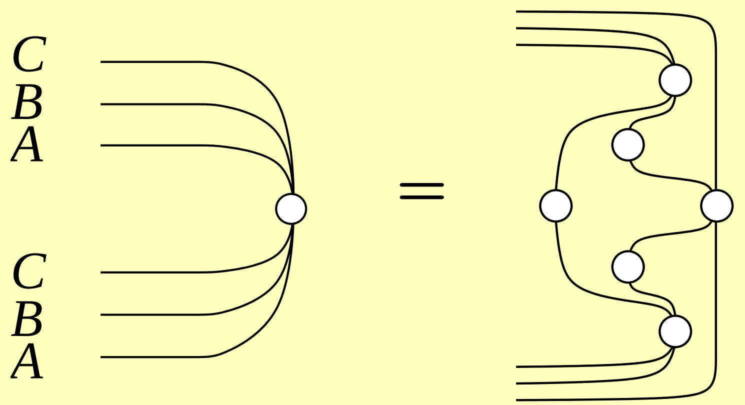
(M3) $(f \otimes g)^{\hat{*}} = f^{\hat{*}} \otimes g^{\hat{*}}$.

(M4) $f = f^{\hat{*}\hat{*}}$.

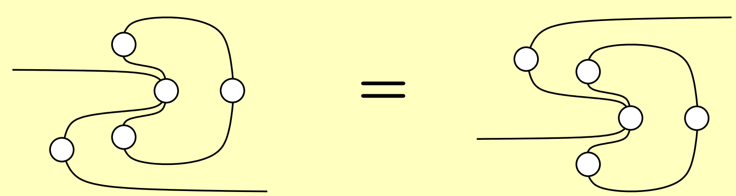
(M5)



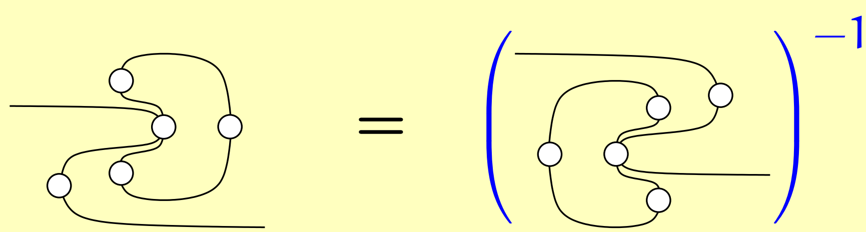
(M6)



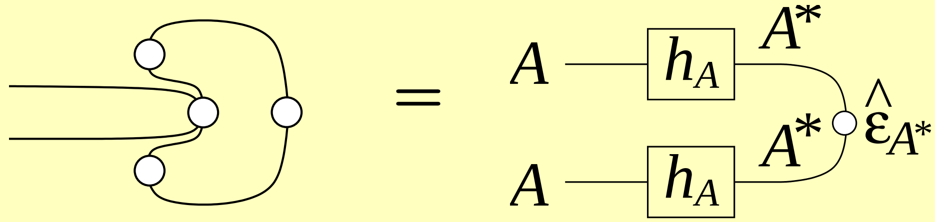
(M7)



(M8)



(M9) $h_A : A \to A^*$ is invertible

(M10) 

Theorem. Axioms (M1)–(M10) are equivalent to autonomy and axioms (A1)–(A8).

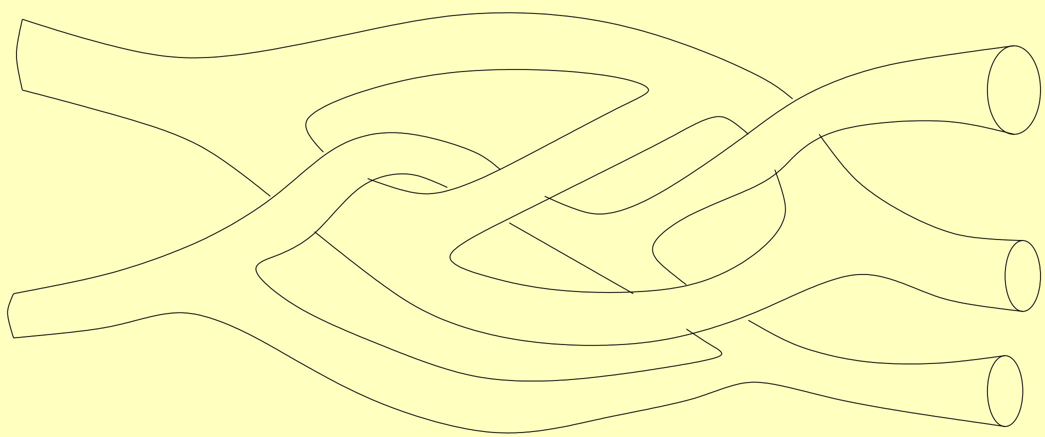
Note.

- Only the last two axioms mention h.
- h is not definable in terms of $\hat{\eta}$ and $\hat{\epsilon}$. However:

Theorem. Every category (without h) satisfying (M1)–(M8) can be fully and faithfully embedded in a category (with h) satisfying (M1)–(M10). **Corollary:** (M9)-(M10) are *conservative* over (M1)–(M8).

An open question

- What are the right coherence conditions for monoidal categories with a chosen Frobenius algebra structure on each object? There is an obvious graphical language.



References

Half-twist maps have been considered before, at varying levels of generality:

[Egger 2010]: in the context of involutive monoidal categories

[Snyder-Tingley 2009]: in the context of Hopf algebras

The end