Strong functors and monads

Kruna Segrt

Laboratoire Jean – Alexandre Dieudonné
Université de Nice – Sophia Antipolis

21 June 2010

Strong functors and monads

Strong functor

Let $\mathcal{E}$ be a symmetric monoidal closed category. Let $\mathcal{A}$ and $\mathcal{B}$ be two $\mathcal{E}$-categories tensored over $\mathcal{E}$. A strong functor (T, σ) consists in giving:

1. A functor $T : \mathcal{A} \longrightarrow \mathcal{B}$;
2. A tensorial strength $\sigma_{X,A} : X \otimes TA \longrightarrow T(X \otimes A)$;
3. Axioms given by the commutativity of the following diagrams:

Strong functors and monads

Unit axiom

$$\begin{array}{ccc} I \otimes TA & \xrightarrow{\sigma_{I,A}} & T(I \otimes A) \\ & {\scriptstyle l_{TA}} \searrow \quad \swarrow {\scriptstyle T(l_A)} & \\ & TA & \end{array}$$

Associativity axiom

$$\begin{array}{ccc} X \otimes Y \otimes TA & \xrightarrow{X \otimes \sigma_{Y,A}} & X \otimes T(Y \otimes A) \\ & {\scriptstyle \sigma_{X \otimes Y, A}} \searrow \quad \swarrow {\scriptstyle \sigma_{X, Y \otimes A}} & \\ & T(X \otimes Y \otimes A) & \end{array}$$

Strong functors and monads

Strong natural transformation

Let $\mathcal{E}$ be a symmetric monoidal closed category. Let $\mathcal{A}$ and $\mathcal{B}$ be two $\mathcal{E}$-categories tensored over $\mathcal{E}$ and let (T_1, σ_1),(T_2, σ_2) be two strong functors such that $T_1, T_2 : \mathcal{A} \longrightarrow \mathcal{B}$.
A strong natural transformation $\Psi : T_1 \longrightarrow T_2$ is given by the following commutative diagram:

$$\begin{array}{ccc}
X \otimes T_1 A & \xrightarrow{\sigma_1} & T_1 (X \otimes A) \\
\Big\downarrow {\scriptstyle X \otimes \Psi_A} & & \Big\downarrow {\scriptstyle \Psi_{X \otimes A}} \\
X \otimes T_2 A & \xrightarrow{\sigma_2} & T_2 (X \otimes A)
\end{array}$$

Strong functors and monads

Lemma

Strong functors and strong natural transformations constitute the 1-cells and 2-cells of a 2-category of $\mathcal{E}$-tensored categories, written **CatStrong**.

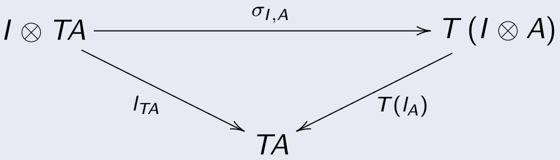
Strong functors and monads

Strong monad

Let $\mathcal{E}$ be a monoidal category. A strong monad (T, μ, η, σ) in a category $\mathcal{E}$ consists in giving:

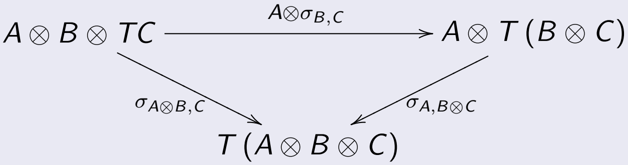
1. A monad (T, μ, η) in a category $\mathcal{E}$;
2. A tensorial strength $\sigma_{A,B} : A \otimes TB \longrightarrow T(A \otimes B)$;
3. Axioms given by the commutativity of the following diagrams:

Unit condition for σ

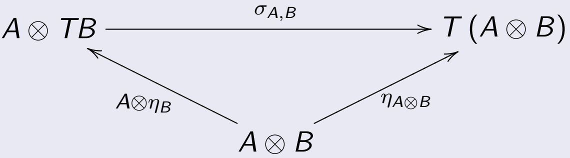


Strong functors and monads

Associativity condition for σ



Strong naturality condition for η



Strong functors and monads

Strong naturality condition for μ

$$
\begin{array}{ccccc}
A \otimes T^2 B & \xrightarrow{\sigma_{A,TB}} & T(A \otimes TB) & \xrightarrow{T(\sigma_{A,B})} & T^2(A \otimes B) \\
\Big\downarrow{\scriptstyle A \otimes \mu_B} & & & & \Big\downarrow{\scriptstyle \mu_{A \otimes B}} \\
A \otimes TB & & \xrightarrow{\qquad\quad \sigma_{A,B} \qquad\quad} & & T(A \otimes B)
\end{array}
$$

Strength and enrichment

Construction

Let $\mathcal{E}$ be a symmetric monoidal closed category. Let $\mathcal{A}$ and $\mathcal{B}$ be two categories tensored over $\mathcal{E}$ and let $(T, \varphi) : \mathcal{A} \to \mathcal{B}$ be a $\mathcal{E}$-functor where $\varphi_{A,B} : \underline{\mathcal{A}}(A, B) \longrightarrow \underline{\mathcal{B}}(TA, TB)$ denotes the enrichment. We define a tensorial strength $\sigma_{X,A} : X \otimes TA \longrightarrow T(X \otimes A)$ by the following commutative diagram:

$$\begin{array}{ccc}
X \otimes TA & \xrightarrow{\sigma_{X,A}} & T(X \otimes A) \\
\downarrow{\scriptstyle \gamma_A \otimes TA} & & \uparrow{\scriptstyle ev_{TA}} \\
\underline{\mathcal{A}}(A, X \otimes A) \otimes TA & \xrightarrow{\varphi_{A, X \otimes A} \otimes TA} & \underline{\mathcal{B}}(TA, T(X \otimes A)) \otimes TA
\end{array}$$

By adjunction, we have $\widehat{\sigma} = \varphi \circ \gamma$.

Strength and enrichment

Similarly, to a tensorial strength $\sigma_{X,A} : X \otimes TA \longrightarrow T(X \otimes A)$ we associate an enrichment $\varphi_{A,B} : \underline{\mathcal{A}}(A,B) \longrightarrow \underline{\mathcal{B}}(TA,TB)$ by the following commutative diagram:

$$
\begin{array}{ccc}
\underline{\mathcal{A}}(A,B) & \xrightarrow{\quad \varphi_{A,B} \quad} & \underline{\mathcal{B}}(TA,TB) \\
\Big\downarrow {\scriptstyle \gamma_{TA}} & & \Big\uparrow {\scriptstyle \underline{\mathcal{B}}(TA,T(ev))} \\
\underline{\mathcal{B}}(TA, \underline{\mathcal{A}}(A,B) \otimes TA) & \xrightarrow{\underline{\mathcal{B}}(TA,\sigma_{\underline{\mathcal{A}}(A,B),A})} & \underline{\mathcal{B}}(TA, T(\underline{\mathcal{A}}(A,B) \otimes A))
\end{array}
$$

By adjunction, we have $\widehat{\varphi} = T(ev) \circ \sigma$.

Strength and enrichment

Lemma

The two constructions are mutually inverse i.e. there is a canonical correspondence between

1. An enrichment of a functor T:
   $\varphi_{A,B} : \underline{\mathcal{A}}(A, B) \longrightarrow \underline{\mathcal{B}}(TA, TB)$
2. A tensorial strength for a functor T:
   $\sigma_{X,A} : X \otimes TA \longrightarrow T(X \otimes A)$

Strength and enrichment

Proposition

Let $\mathcal{E}$ be a symmetric monoidal closed category. Given two categories $\mathcal{A}$ and $\mathcal{B}$ tensored over $\mathcal{E}$ and a functor $T : \mathcal{A} \longrightarrow \mathcal{B}$, the following conditions are equivalent:

1. A functor T extends to a strong functor (T, σ)
2. A functor T extends to a $\mathcal{E}$-functor (T, φ)

Strength and enrichment

Proposition

Let $\mathcal{E}$ be a monoidal category. Let $\mathcal{A}$ and $\mathcal{B}$ be two $\mathcal{E}$-categories and $T_1, T_2 : \mathcal{A} \rightrightarrows \mathcal{B}$ two $\mathcal{E}$-functors. Given a natural transformation $\Psi : T_1 \longrightarrow T_2$, the following conditions are equivalent:

1. A natural transformation Ψ extends to a strong natural transformation (Ψ, σ)
2. A natural transformation Ψ extends to a $\mathcal{E}$-natural transformation (Ψ, φ)

Strength and enrichment

Theorem

A 2-category of strong functors and strong natural transformations of tensored $\mathcal{E}$-categories, called **CatStrong** is 2-isomorphic to a 2-category of $\mathcal{E}$-functors and $\mathcal{E}$-natural transformations of tensored $\mathcal{E}$-categories, called $\mathcal{E}$-**Cat**.

Strength and enrichment

Corollary

Let $\mathcal{C}$ be a monoidal category. Given a monad (T, μ, η) in a category $\mathcal{C}$, the following conditions are equivalent:

1. A monad (T, μ, η) extends to a strong monad (T, μ, η, σ)
2. A monad (T, μ, η) extends to a $\mathcal{E}$-monad (T, μ, η, φ)

Strong monad and Morita theory

Proposition

Let $\mathcal{E}$ be a symmetric monoidal closed category with equalizers and (T, μ, η, φ) an enriched monad over $\mathcal{E}$. Then the category $\mathbf{Alg}_T$ of T-algebras is canonically enriched over $\mathcal{E}$.

Proposition

Let $\mathcal{E}$ be a symmetric monoidal closed category with coequalizers and (T, μ, η, σ) a strong monad on $\mathcal{E}$. Then the category $\mathbf{Alg}_T$ of T-algebras is tensored over $\mathcal{E}$.

Strong monad and Morita theory

Proposition

Let (T, μ, η, σ) be a strong monad. Then the object $T(I)$ has a structure of a monoid, namely it may be identified with $\mathcal{A}lg_T(T(I), T(I))$.

Lemma

For each monoid M the endofunctor $- \otimes M$ has a canonical structure of a strong monad.

Proposition

For each strong monad (T, μ, η, σ) there is a canonical map of strong monads $- \otimes T(I) \to T$.
This map is an isomorphism if and only if the monad T is induced by a monoid.

Strong monad and Morita theory

Definition

A monoidal (Quillen) model category $\mathcal{C}$ is a category which is at once:

1. A closed symmetric monoidal category
2. A closed Quillen model category
3. Such that the pushout-product axiom of Hovey is satisfied

Strong monad and Morita theory

Theorem

Let $\mathcal{E}$ be a monoidal model category cofibrantly generated with cofibrant unit. Let $\mathcal{A}lg_T$ be a category of T-algebras and assume that $\mathcal{A}lg_T$ admits a model structure.

Consider a strong monad (T, μ, η, σ) such that

1. The tensorial strength $\sigma_{X,Y} : X \otimes TY \to T(X \otimes Y)$ is a weak equivalence for X,Y cofibrant in $\mathcal{E}$
2. The unit $\eta : I \to TI$ is a cofibration in $\mathcal{E}$

Then the monad morphism $\omega : - \otimes T(I) \to T$ induces a Quillen equivalence $\omega! : \mathcal{A}lg_T \rightleftarrows Mod_{T(I)} : \omega^*$.

Strong monad and Morita theory

Exemple

Suppose that $\mathcal{E}$ is a category of pointed simplicial sets.
Then a simplicial (reduced) Γ-ring gives rise to strong monad on pointed simplicial sets. If the underlying simplicial Γ-set is cofibrant in Bousfield-Friedlander sense, then the strong monad satisfies the axiom of our theorem and we recover a result of Stefan Schwede.

Bibliography

- **Kock, Anders**, *Strong functors and monoidal monads*, Arch. Math. (Basel), **23**, (1972), 113–120.
- **Kock, Anders**, *Monads on symmetric monoidal closed categories*, Arch. Math. (Basel), **21**, (1970), 1–10.
- **Schwede, Stefan**, *Stable homotopy of algebraic theories*, Topology, **40**, (2001),1, 1–41.
- **Lack, Stephen and Street, Ross**, *The formal theory of monads. II*, J. Pure Appl. Algebra, **175**, (2002),1-3, 243–265.