The Unique Solution Property

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Framework

Bishop-style constructive mathematics without choice.
Intuitionistic logic. Suitable fragment of CZF. Unique Choice.
No non-unique countable choice, let alone dependent choice.

Completions without sequences; real numbers: Dedekind cuts.
Continuity: uniform continuity on compact domains.
Compactness: total boundedness plus completeness.

Setting

Let Y be a metric space, and $F : Y \to \mathbb{R}$ uniformly continuous.

If Y is totally bounded, then $\inf F$ can be computed;
whence we may tacitly assume that $\inf F = 0$.

If Y is compact, can one locate a minimum of F?
I.e., can one find a point of Y at which F attains its infimum?

Heuristics: constructive solutions are continuous in the parameters.
To rule out any potential discontinuity, (uniform) uniqueness helps.

Variants of Uniqueness

Suppose that inf $F = 0$. We agree that $y, y' \in Y$ and $\varepsilon, \delta > 0$.

Any such F has *uniformly at most one minimum* if

$$\forall\delta\,\exists\varepsilon\,\forall y, y' \, \left[F(y) < \varepsilon \wedge F(y') < \varepsilon \Rightarrow d(y, y') < \delta\right]$$

or, equivalently,

$$\forall\delta\,\underline{\exists\varepsilon\,\forall y, y'} \, \left[d(y, y') \geqslant \delta \Rightarrow F(y) \geqslant \varepsilon \vee F(y') \geqslant \varepsilon\right] .$$

The Meta-Theorem of Unique Existence

Let Y be complete, and F uniformly continuous.
If $\inf F = 0$ *and F has uniformly at most one minimum,*
then there is $y \in Y$ with $F(y) = 0$.

The meta-theorem can be extended to the case of an equation

$$F(x, y) = 0$$

with a parameter x varying over a metric space X. In this case there even is a continuous function $f : X \to Y$ with $F(x, f(x)) = 0$.

As a proof paradigm this has a considerable history:

Lifshitz 1971, Gelfond 1972, Kreinovich 1979, Bridges 1980, Aczel 1987, Ko 1986, Kohlenbach 1993, Weihrauch 2000, Oliva 2002, Kohlenbach-Oliva 2003, Bauer-Taylor 2005, Brattka 2008, ...

(to mention for each author only the first printed occurrence)

“Logicians, is there a meta-theorem to explain it?” (Beeson 1985)

The essence of the meta-theorem

— can be traced back to Russian recursive mathematics;

— has proved productive in constructive/computable analysis;

— stood right at the beginnings of the so-called proof mining;

— has recently reoccurred in abstract Stone duality.

The uniqueness hypothesis helps to locate the minimum above any "pure existence proof" (e.g. by Bolzano-Weierstraß or WKL) tied together with a fragment of the Law of Excluded Middle.

Sequential Proof of the Meta-Theorem (folklore)

Since $\inf F = 0$ one can choose (!) a sequence (y_n) in Y with

$$F(y_n) < 1/n,$$

which is Cauchy for F has uniformly at most one minimum.
Since Y is complete, (y_n) has a limit y in Y, for which $F(y) = 0$.
Uniform uniqueness is essential to get a *Cauchy* sequence.

Even if Y fails to be complete, the given data are converted—by countable choice—into an element of the completion of Y. This gave us the clue of how to get by without choice (Sch. 2010).

Doing Without Countable Choice

The completion of Y now is the set $\hat{Y}$ of locations (Richman 2000).

Similar methods to define completions without sequences:
Mulvey 1979, Burden and Mulvey 1979, Stolzenberg 1988, Vickers 2005, Fox 2005, Palmgren 2007, . . .

Let $\mathbb{R}$ denote the set of Dedekind reals: that is, located cuts in $\mathbb{Q}$.

A *location* on Y is a function $f : Y \to \mathbb{R}$ with $\inf f = 0$ and

$$\left| f(y) - f\left(y'\right) \right| \leqslant d\left(y, y'\right) \leqslant f(y) + f\left(y'\right) .$$

The set $\hat{Y}$ of all locations on Y is a metric space with metric

$$d(f, g) = \sup |f - g| = \inf (f + g) .$$

There is the isometric embedding

$$Y \hookrightarrow \widehat{Y}\,,\ y \mapsto \widehat{y} = d\,(y,\,\cdot\,)\ ,$$

along which (each point of) Y is identified with its image in $\widehat{Y}$.

Needless to say, $\widehat{Y}$ is complete; and so is $\mathbb{R}$ for $\mathbb{R} \cong \widehat{\mathbb{Q}}$.

Every location measures the distance between itself and the points:

$$d\,(f, \widehat{y}) = f\,(y)\ .$$

The problem indeed provides us with its own solution (Sch. 2010):

$$f\,(y) = \lim_{F(y') \to 0} d\left(y, y'\right)\ ,\ \ f = \widehat{z}\ \ \Rightarrow\ \ F(z) = 0\,.$$

But *why* does uniform uniqueness help at all to find the solution?

An Equivalent of Completeness

Definition A metric space Y has the *unique solution property* (USP) if for every uniformly continuous $F : Y \to \mathbb{R}$ with $\inf F = 0$ and uniformly at most one minimum there is $y \in Y$ with $F(y) = 0$.

The meta-theorem of unique existence thus says that every complete metric space has the USP. The converse is also valid:

Theorem 1 *A metric space Y has the USP iff Y is complete.*

In fact, Y is complete already if every location f on Y attains its infimum 0 at a point y of Y, for which $f = \widehat{y}$ because $d(f, \widehat{y}) = f(y)$.

Proof for completions with Cauchy sequences:

If (y_n) is a Cauchy sequence in Y, then

$$f(y) = \lim_{n\to\infty} d(y, y_n)$$

defines a location f on Y such that

$$f(y) = 0 \Leftrightarrow \lim_{n\to\infty} y_n = y\,.$$

Moduli of convergence and of uniqueness correspond to each other.

"Cauchy sequence" and uniform uniqueness have the same form:

$$\forall\delta\,\exists N\,\forall k, k' \,\Big[k \geqslant N \wedge k' \geqslant N \;\Rightarrow\; d(y_k, y_{k'}) < \delta\Big]$$

$$\forall\delta\,\exists\varepsilon\,\forall y, y' \,\Big[F(y) < \varepsilon \wedge F(y') < \varepsilon \;\Rightarrow\; d(y, y') < \delta\Big]$$

Metric Spaces as Categories (Lawvere 1973)

Each (generalised quasi & pseudo) metric space X is an $\mathbb{R}$-*category*:

$$\text{objects: } a \in X \qquad \text{morphisms: } X(a,b) \in [0,\infty]$$

i.e. X is a strong category valued in the closed category $\mathbb{R}$ with

$$\text{objects: } r \in [0,\infty] \qquad \text{morphisms: } r \geqslant s$$

In $\mathbb{R}$ we have $r \otimes s = r + s$ with unit 0; whence in X we have

$$X(a,b) \otimes X(b,c) \to X(a,c) \;\leftrightsquigarrow\; X(a,b) + X(b,c) \geqslant X(a,c)\,,$$

$$0 \to X(a,a) \;\leftrightsquigarrow\; 0 \geqslant X(a,a)\,.$$

Let X, Y be metric spaces with metrics X, Y.
An $\mathbb{R}$-*functor* is a map $f : X \to Y$ with

$$Y(f(a), f(b)) \leqslant X(a, b) \,.$$

An $\mathbb{R}$-*bimodule* is a map $F : Y \times X \to [0, \infty]$ with

$$Y(y', y) + F(y, x) \geqslant F(y', x) \,, \quad F(y, x) + X(x, x') \geqslant F(y, x') \,.$$

Each $\mathbb{R}$-functor $f : X \to Y$ *induces* two $\mathbb{R}$-bimodules f_*, f^*:

$$f_*(y, x) = Y(y, f(x)) \,, \quad f^*(x, y) = Y(f(x), y) \,,$$

which form an *adjoint pair* (f_*, f^*) of $\mathbb{R}$-bimodules:

$$X(x, x') \geqslant \inf_{y \in Y} [f^*(x, y) + f_*(y, x')] \,, \ \inf_{x \in X} [f_*(y, x) + f^*(x, y')] \geqslant Y(y, y') \,.$$

Theorem 2 *(Lawvere 1973)*
A metric space Y is complete if and only if every adjoint pair (f_*, f^*) of $\mathbb{R}$-bimodules is induced by an $\mathbb{R}$-functor $f : X \to Y$.

The proof "is" the sequential proof of "USP $\Leftrightarrow$ completeness".

Can also be proved with locations in place of Cauchy sequences:

If X is complete, then (f_*, f^*) is induced by f where

$$y \mapsto \lim_{f_*(y',x)+f^*(x,y')\to 0} Y(y, y')$$

defines the location on Y that locates $f(x)$.

Conversely, if λ is a location on Y, and the adjoint pair

$$f^*(y) = \lim_{\lambda(y')\to 0} Y(y', y)\,, \quad f_*(y) = \lim_{\lambda(y')\to 0} Y(y, y')$$

is induced by $z \in Y$, then $\lambda = \widehat{z}$.