Goursat categories and the 3×3 Lemma

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Goursat cats $\iff$ pp w/ Goursat pushouts

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$\mathbf{3} \times \mathbf{3}$ Lemma

$\therefore$ $\mathcal{C}$ regular

$\mathcal{C}$ Goursat iff the $\mathbf{3} \times \mathbf{3}$ Lemma holds

Helpful tool

Goursat pos: • helpful tool

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• previously used in

Thm (Gran, R, 2008)

If $\mathbb{C}$ is a Goursat category with coequalisers, then:

1. $\mathbf{Gpd}(\mathcal{C})$ is a full reflective subcategory of $\mathbf{RG}(\mathcal{C})$
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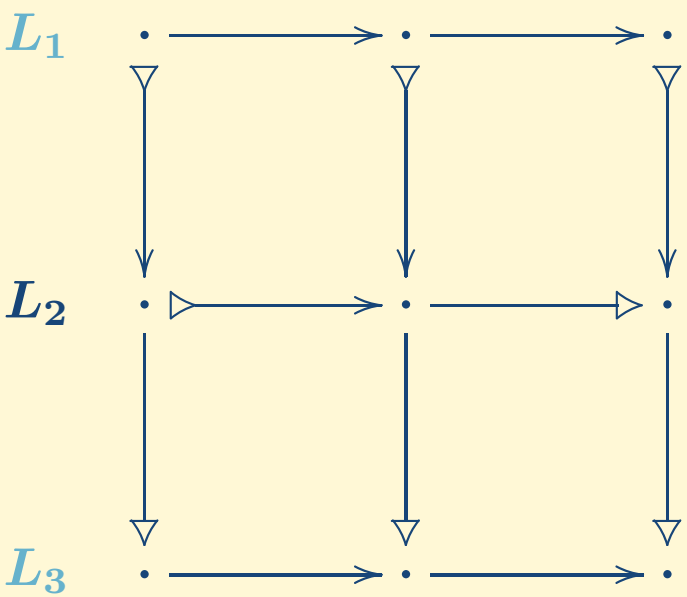
The (denormalized) 3×3 Lemma

3×3 Lemma



$\exists\ \mathbf{0}$

exact: short exact seq

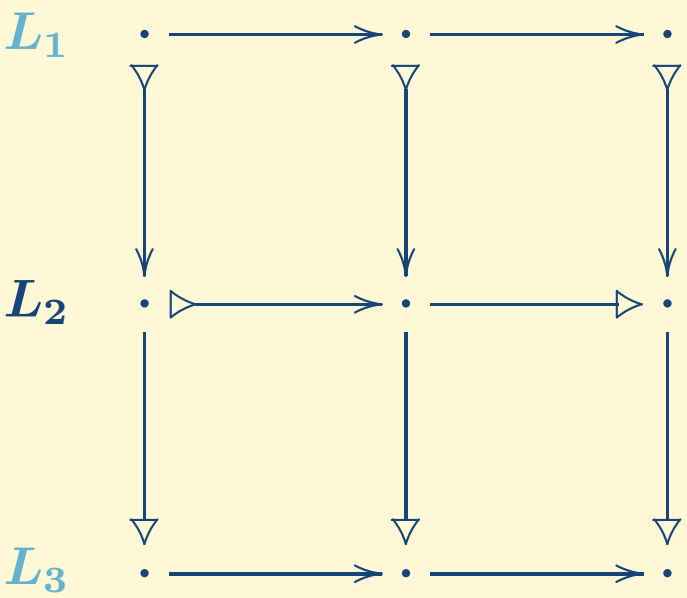
columns exact

L_2 exact

L_1 exact $\Leftrightarrow$ L_3 exact

The (denormalized) 3×3 Lemma

denormalized $\quad 3 \times 3$ Lemma



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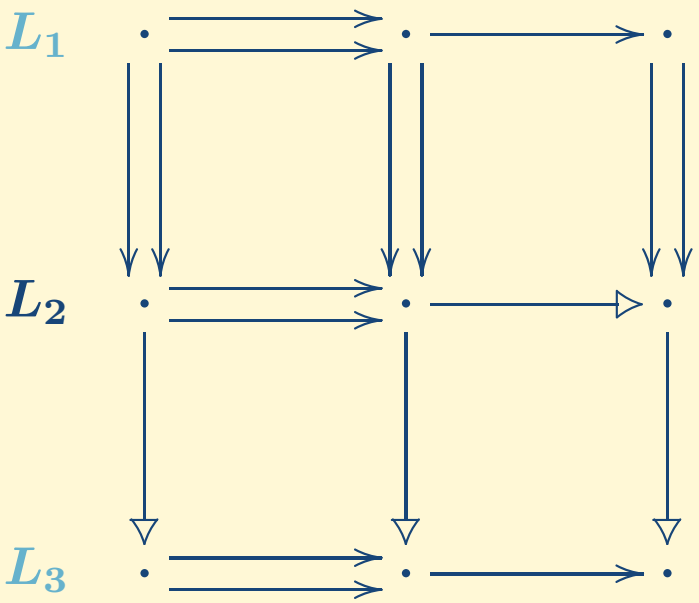

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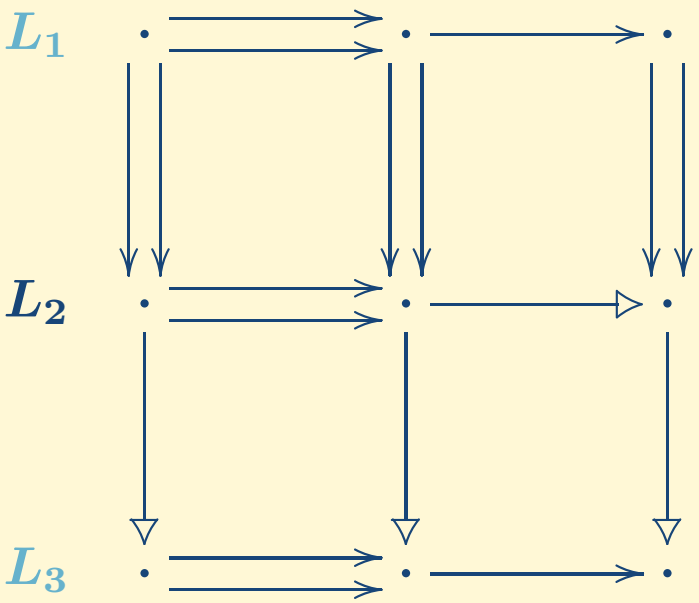

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(6 out of 5)

Goursat categories

Definition

$\mathcal{C}$ (regular) Goursat: $RSR = SRS$, $\forall R, S$ equiv

($\mathbf{3}$-permutable)

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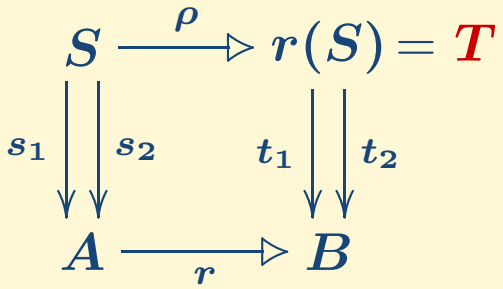

S equiv $\Rightarrow$ T equiv

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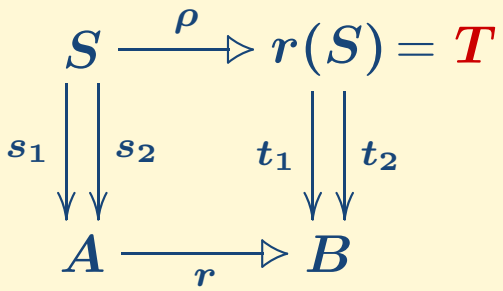

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(S reflexive & symmetric $\Rightarrow$ T reflexive & symmetric)

Definition

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OR (Carboni, Kelly, Pedicchio)

$$\begin{array}{ccc} S & \xrightarrow{\rho} & r(S) = T \\ {\scriptstyle s_1}\downarrow\downarrow{\scriptstyle s_2} & & {\scriptstyle t_1}\downarrow\downarrow{\scriptstyle t_2} \\ A & \xrightarrow[r]{} & B \end{array}$$

S equiv $\Rightarrow$ T equiv

(S reflexive & symmetric $\Rightarrow$ T reflexive & symmetric)

S equiv $\Rightarrow$ T transitive $(T \cdot T = T)$

Examples

Examples:

Mal'cev cats: $\mathbf{Gp}$, $\mathbf{Rng}$, $\mathbf{Alg}(\mathbb{T})$, $\mathbb{T}$ algebraic theory w/ group op

or $\mathbb{T}$ w/ Mal'cev operation

$$p(x, y, y) = x, \quad p(x, x, y) = y$$

$\mathbf{Heyt}$, $(\,\mathbf{topos}\,)^{op}$, $\cdots$

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non Mal'cev cats: implication algebras (Mitschke)

$\exists$ binary op $\to$ and constant 1 sth

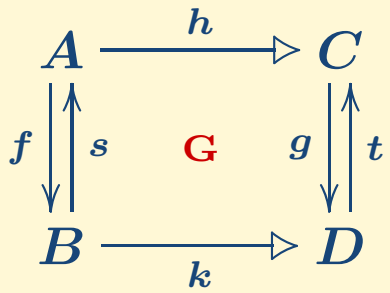
$(x \to y) \to x = x, \quad (x \to y) \to y = (y \to x) \to x$

$x \to (y \to z) = y \to (x \to z), \quad 1 \to x = x$

right-complemented semigroups (Hagemann, Mitschke)

Goursat pushouts

Goursat pushout:



$$f \cdot s = 1$$
$$g \cdot t = 1$$
$$k \cdot f = g \cdot h$$
$$h \cdot s = t \cdot k$$

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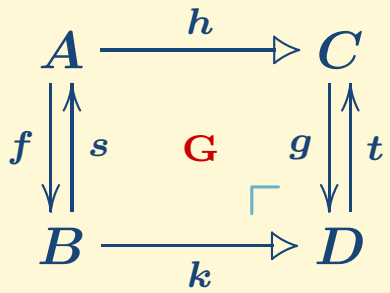

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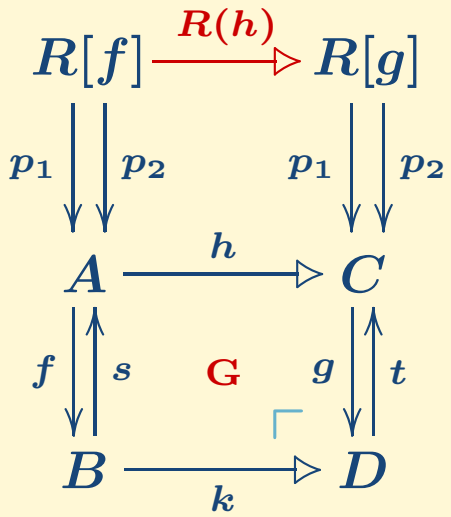

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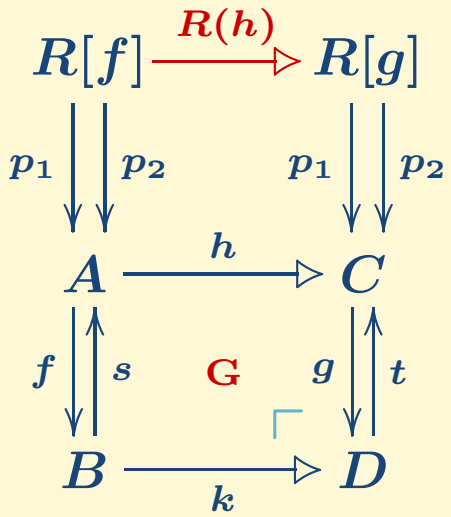

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Goursat po $\neq$ regular po

Bourn

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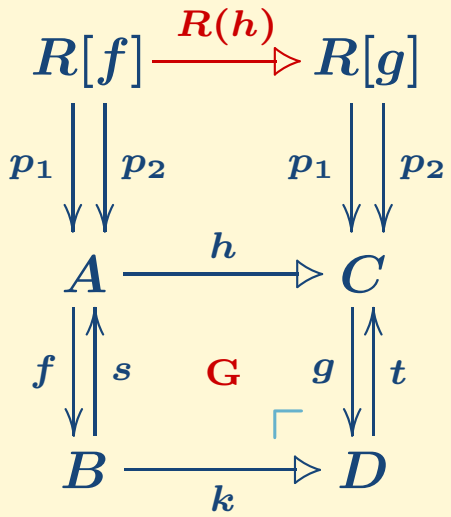

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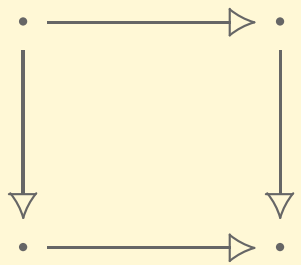

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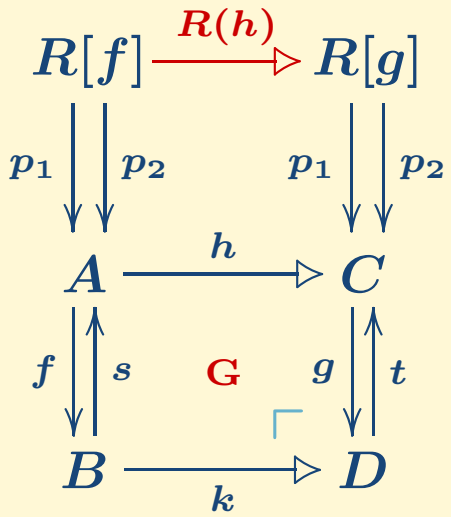

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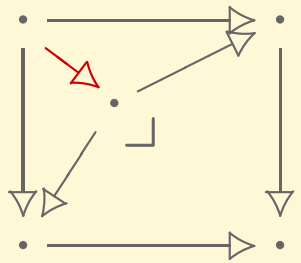

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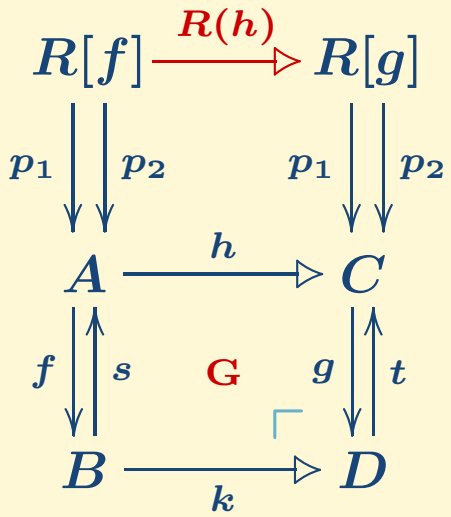

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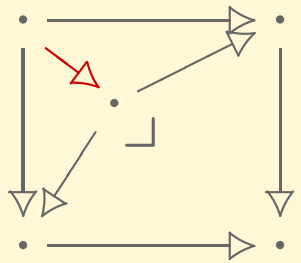

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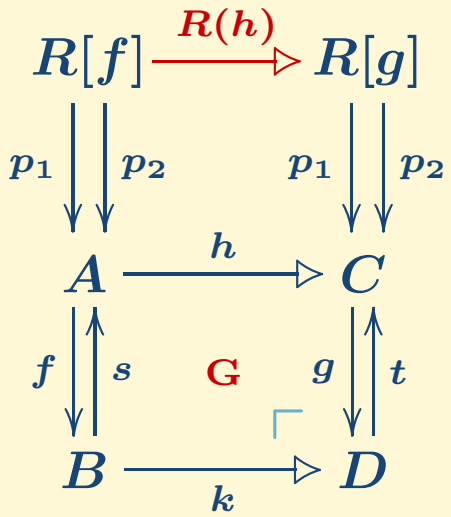

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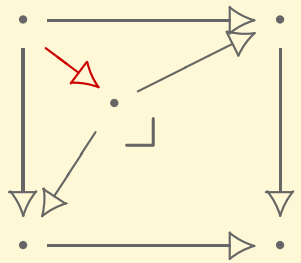

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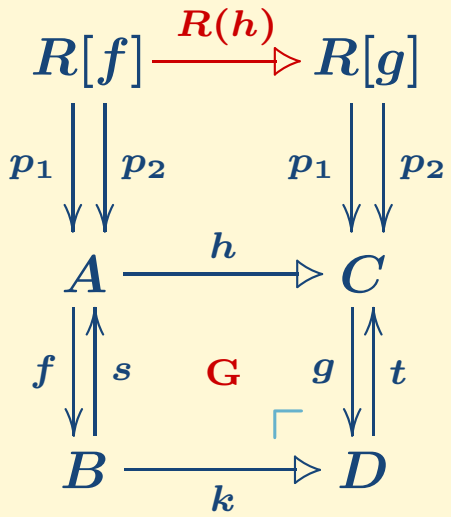

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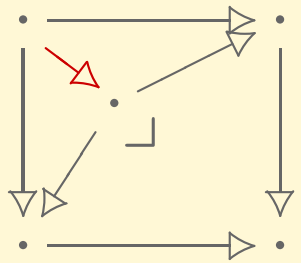

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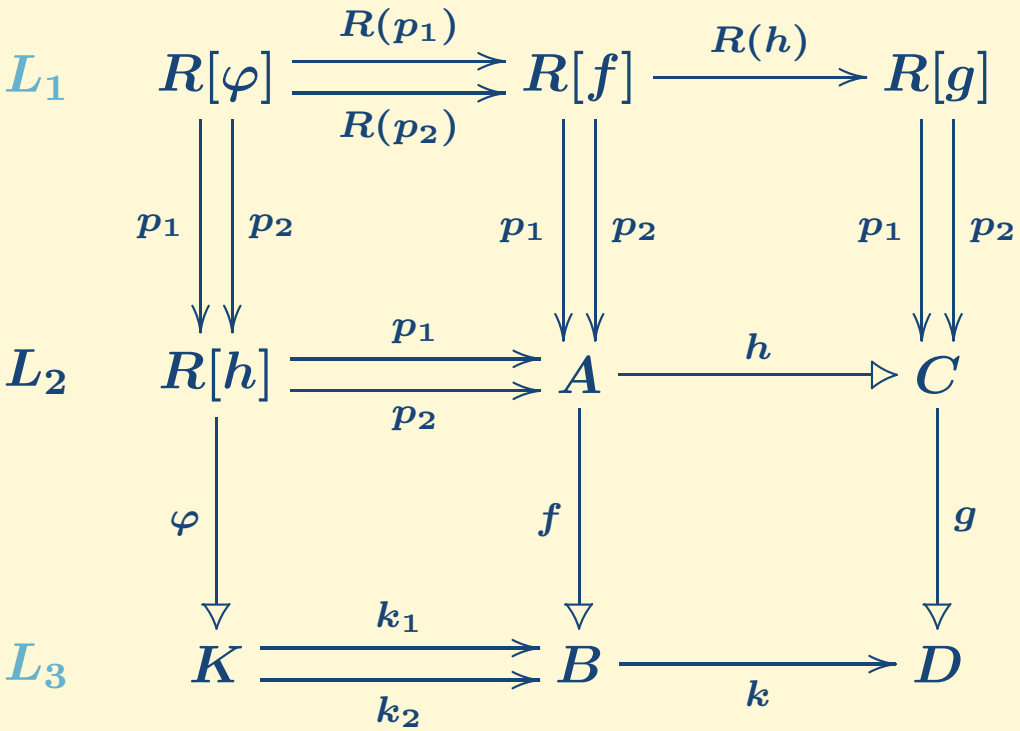
The 3×3 Lemma

Weaker conditions



$\mathbf{3} \times \mathbf{3}$ Lemma: L_1 exact $\Leftrightarrow$ L_3 exact

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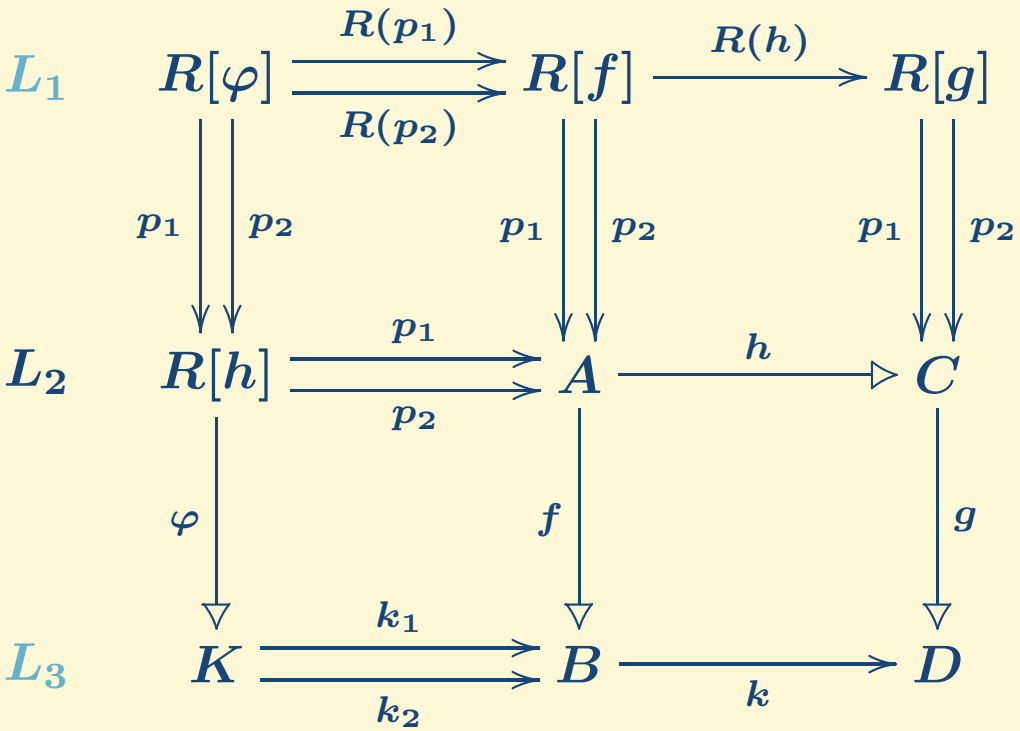

$W_{1\Rightarrow 3}$: L_1 exact $\Rightarrow$ L_3 exact

$+$ $W_{3\Rightarrow 1}$: L_1 exact $\Leftarrow$ L_3 exact

3×3 Lemma: L_1 exact $\Leftrightarrow$ L_3 exact

Equivalent Goursat properties

Thm: $\mathcal{C}$ regular. TFAE:

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Proof: $(a) \Rightarrow (b) \Rightarrow (c)$ & (d) ✓

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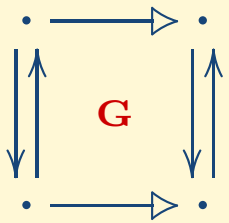

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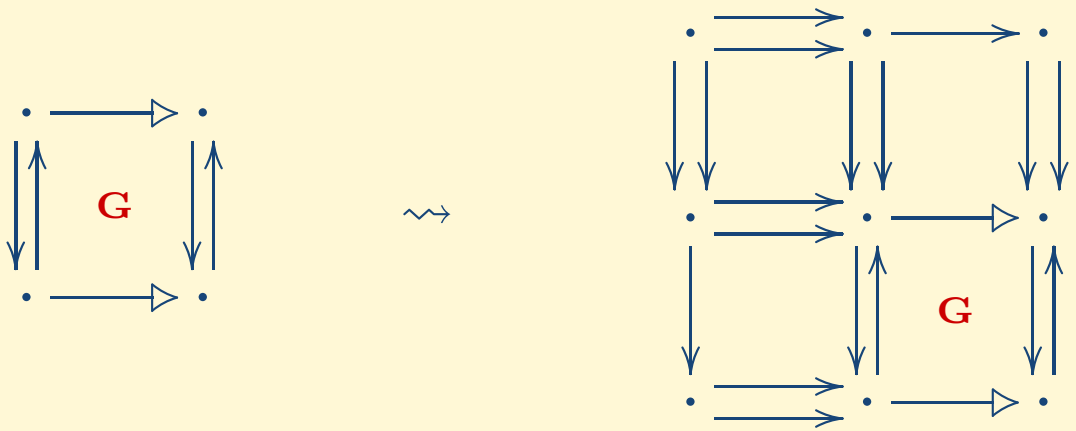

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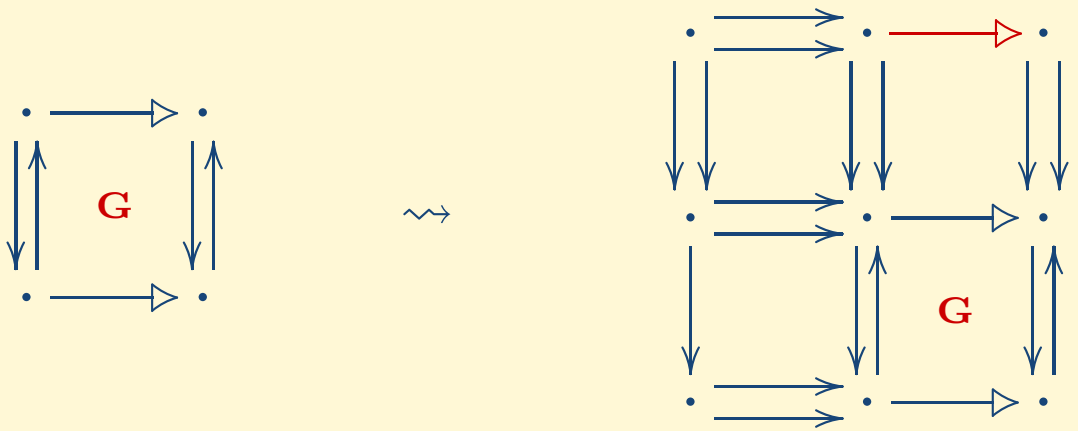

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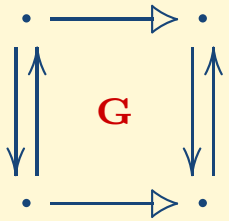
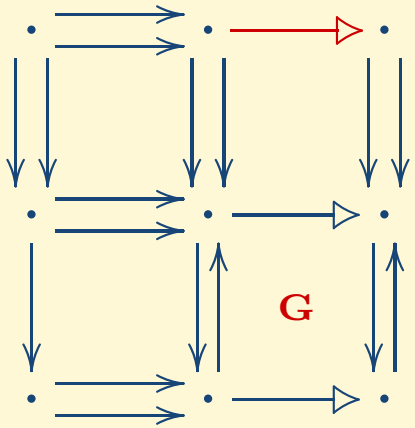
$\forall$ 

Goursat po

$\rightsquigarrow$ 

3-permutable varieties

Characterisation

$\mathcal{C}$ $\mathbf{3}$-permutable variety $\Leftrightarrow$ $\mathcal{C}$ Goursat category

$RSR = SRS$

Characterisation

$\mathcal{C}$ $\mathbf{3}$-permutable variety $\Leftrightarrow$ $\mathcal{C}$ Goursat category

$RSR = SRS$

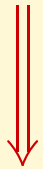
$\Leftrightarrow$ $\mathbb{T}$ contains quaternary ops p and q sth

$$\text{(I)} \begin{cases} p(x,y,y,z) = x \\ q(x,y,y,z) = z \\ p(x,x,y,y) = q(x,x,y,y) \end{cases}$$

Characterisation

$\mathcal{C}$ $\mathbf{3}$-permutable variety $\Leftrightarrow$ $\mathcal{C}$ Goursat category

$RSR = SRS$



Direct proof w/ Goursat pos

$\Leftrightarrow$ $\mathbb{T}$ contains quaternary ops p and q sth

$$\text{(I)} \begin{cases} p(x,y,y,z) = x \\ q(x,y,y,z) = z \\ p(x,x,y,y) = q(x,x,y,y) \end{cases}$$

Direct proof

$\exists$ p and q sth (I) $\begin{cases} p(x,y,y,z) = x, \quad q(x,y,y,z) = z \\ p(x,x,y,y) = q(x,x,y,y) \end{cases}$?

Direct proof

$\exists$ p and q sth (I) $\begin{cases} p(x,y,y,z) = x, \quad q(x,y,y,z) = z \\ p(x,x,y,y) = q(x,x,y,y) \end{cases}$?

$\boldsymbol{X}$: free algebra on one element

Direct proof

$\exists \quad p$ and q sth (I) $\begin{cases} p(x,y,y,z) = x, \quad q(x,y,y,z) = z \\ p(x,x,y,y) = q(x,x,y,y) \end{cases}$?

X: free algebra on one element

$$\begin{array}{ccc} X+X+X+X & \xrightarrow{1+\nabla+1} & X+X+X \\ {\scriptstyle \nabla+\nabla}\downarrow\uparrow{\scriptstyle i_2+i_1} & & {\scriptstyle \nabla_3}\downarrow\uparrow{\scriptstyle i_2} \\ X+X & \xrightarrow[\nabla]{} & X \end{array}$$

Direct proof

$\exists \quad p$ and q sth (I) $\begin{cases} p(x,y,y,z) = x, \quad q(x,y,y,z) = z \\ p(x,x,y,y) = q(x,x,y,y) \end{cases}$?

X: free algebra on one element

$$\begin{array}{ccc} X+X+X+X & \xrightarrow{1+\nabla+1} & X+X+X \\ {\scriptstyle \nabla+\nabla}\downarrow\uparrow{\scriptstyle i_2+i_1} & G & {\scriptstyle \nabla_3}\downarrow\uparrow{\scriptstyle i_2} \\ X+X & \xrightarrow[\nabla]{} & X \end{array}$$

Direct proof

$\exists$ p and q sth (I) $\begin{cases} p(x,y,y,z) = x, \quad q(x,y,y,z) = z \\ p(x,x,y,y) = q(x,x,y,y) \end{cases}$?

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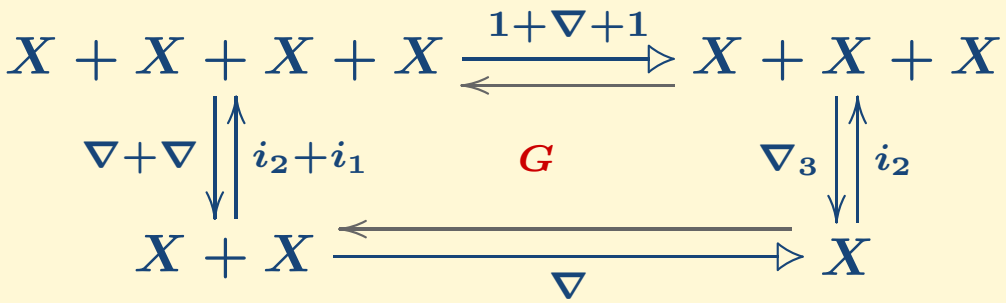

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Goursat po

Direct proof

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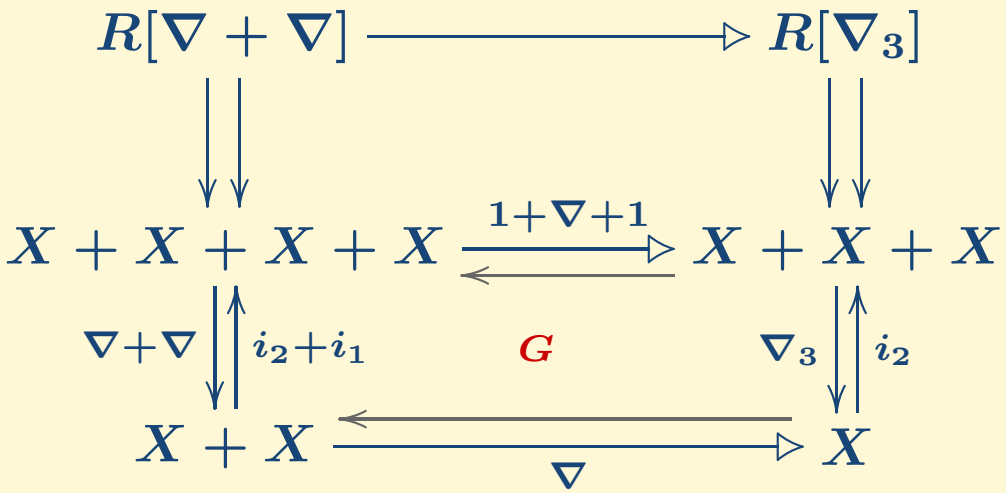

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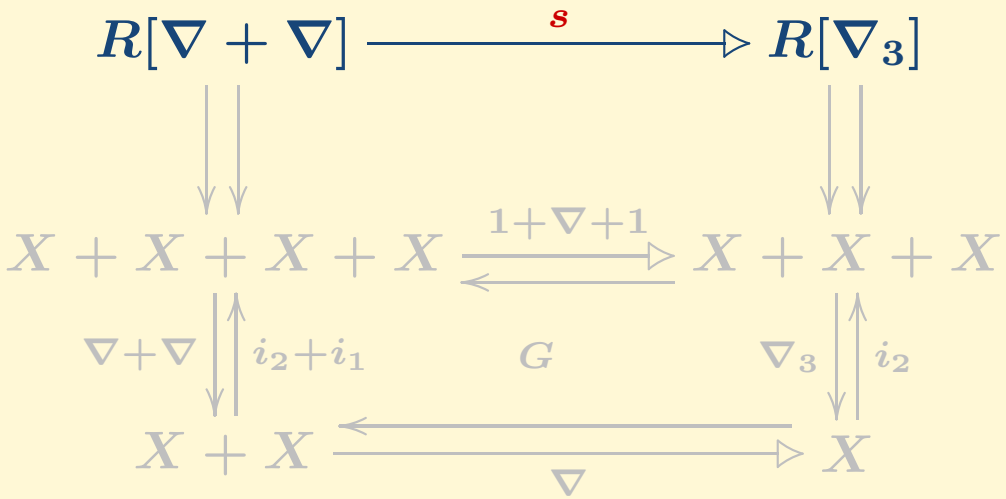

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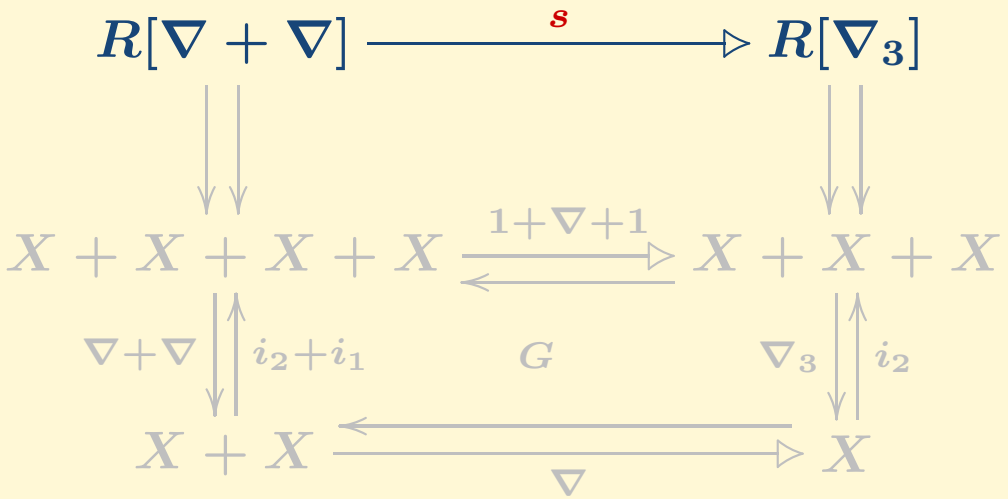

Goursat po

$p_1(x,y,z) = x, \quad p_3(x,y,z) = z \quad \Rightarrow \quad (p_1, p_3) \in R[\nabla_3]$

Direct proof

$\exists$ p and q sth (I) $\begin{cases} p(x,y,y,z) = x, \quad q(x,y,y,z) = z \\ p(x,x,y,y) = q(x,x,y,y) \end{cases}$?

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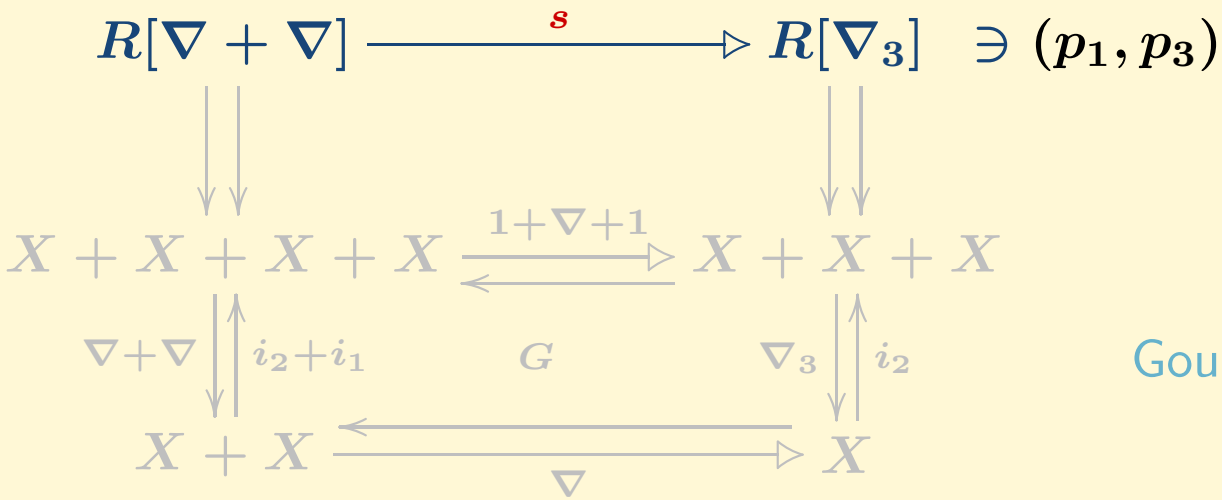

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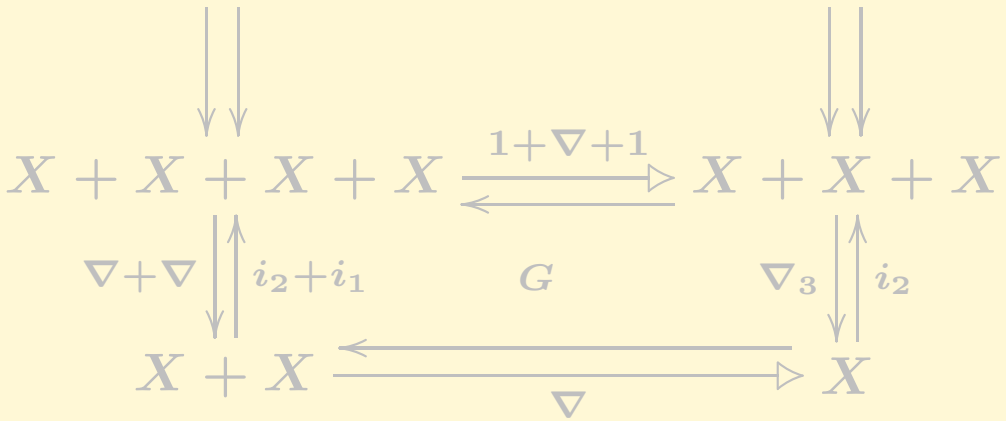
$\boldsymbol{X}$: free algebra on one element

$$\exists\ (p,q) \in\ R[\nabla + \nabla] \xrightarrow{\ s\ } R[\nabla_3]\ \ni (p_1, p_3)$$



Goursat po

$p_1(x,y,z) = x, \quad p_3(x,y,z) = z \quad \Rightarrow \quad (p_1, p_3) \in R[\nabla_3]$

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$$\begin{array}{ccccc} X+X+X+X & \underset{}{\overset{1+\nabla+1}{\rightleftarrows}} & X+X+X & & \\ {\scriptstyle \nabla+\nabla}\downarrow\uparrow{\scriptstyle i_2+i_1} & G & {\scriptstyle \nabla_3}\downarrow\uparrow{\scriptstyle i_2} & & \\ X+X & \underset{\nabla}{\rightleftarrows} & X & & \end{array}$$

Goursat po

$p_1(x,y,z) = x, \quad p_3(x,y,z) = z \quad \Rightarrow \quad (p_1,p_3) \in R[\nabla_3]$

$s(p,q) = (p_1,p_3) \quad \Rightarrow \quad p(x,y,y,z) = x, \quad q(x,y,y,z) = z$

Direct proof

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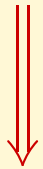
Quasivarieties

$\mathcal{C}$ $\mathbf{3}$-permutable variety $\Leftrightarrow$ $\mathcal{C}$ Goursat category

$RSR = SRS$



Direct proof w/ Goursat pos

$\Leftrightarrow$ $\mathbb{T}$ contains quaternary ops p and q sth

$$\text{(I)} \begin{cases} p(x,y,y,z) = x \\ q(x,y,y,z) = z \\ p(x,x,y,y) = q(x,x,y,y) \end{cases}$$

Quasivarieties

$\mathcal{C}$ $\mathbf{3}$-permutable $\Leftrightarrow$ $\mathcal{C}$ Goursat category

quasi-variety

$RSR = SRS$

$\Downarrow$ Direct proof w/ Goursat pos

$\Leftrightarrow$ $\mathbb{T}$ contains quaternary ops p and q sth

$$\text{(I)} \begin{cases} p(x,y,y,z) = x \\ q(x,y,y,z) = z \\ p(x,x,y,y) = q(x,x,y,y) \end{cases}$$