Equivariant Homotopy Theory for Representable Orbifolds

Dorette Pronk[1] Laura Scull[2]

[1]Dalhousie University

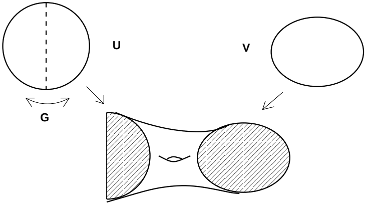
[2]Fort Lewis College

June 26, 2010

Outline

What is an orbifold?

- Orbifolds form a generalization of manifolds.
- Local charts are of the form U/G for some finite group acting on an open set $U \subseteq \mathbb{R}^n$ via diffeomorphisms, $\rho_G \colon G \to \text{Diffeo}\,(U)$,

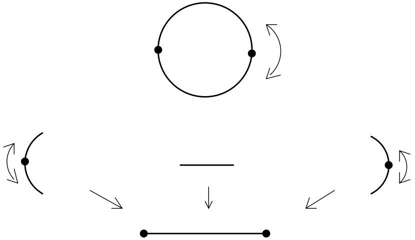


  and they are locally compatible (expressed in terms of embeddings of charts).
- An orbifold is a paracompact Hausdorff space with an equivalence class of orbifold atlases.

Examples

1. Manifolds (no non-trivial isotropy groups)
2. A *global quotient* of a properly discontinuous group G acting on a manifold M

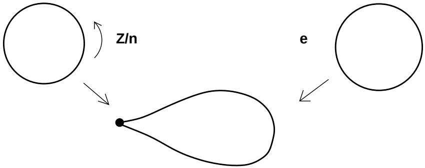
Eg $M = S^1$ with $G = \mathbb{Z}/2$ action



The orbifold consists of the orbit space M/G together with the data about the isotropy groups

Examples

3 (Thurston) Teardrop orbifold:



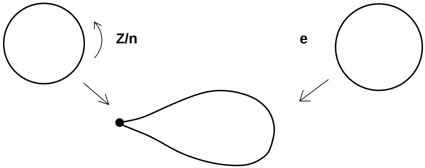
This does not come from a properly discontinuous group action on a manifold.

4 Let L be a compact Lie group, which acts on a manifold X. If the isotropy groups are finite, the orbit space X/L is an orbifold. If an orbifold can be described in this way it is called representable.

5 The teardrop orbifold can be obtained by S^1 acting on S^3 via $\lambda[z_1, z_2] = [\lambda^n z_1, z_2]$

Examples

3 (Thurston) Teardrop orbifold:



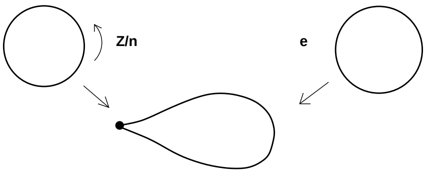
This does not come from a properly discontinuous group action on a manifold.

4 Let L be a compact Lie group, which acts on a manifold X. If the isotropy groups are finite, the orbit space X/L is an orbifold. If an orbifold can be described in this way it is called representable.

5 The teardrop orbifold can be obtained by S^1 acting on S^3 via $\lambda[z_1, z_2] = [\lambda^n z_1, z_2]$

Examples

3 (Thurston) Teardrop orbifold:



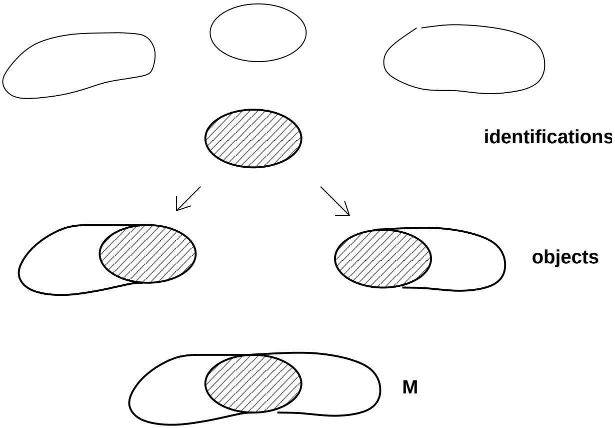
This does not come from a properly discontinuous group action on a manifold.

4 Let L be a compact Lie group, which acts on a manifold X. If the isotropy groups are finite, the orbit space X/L is an orbifold. If an orbifold can be described in this way it is called representable.

5 The teardrop orbifold can be obtained by S^1 acting on S^3 via $\lambda[z_1, z_2] = [\lambda^n z_1, z_2]$

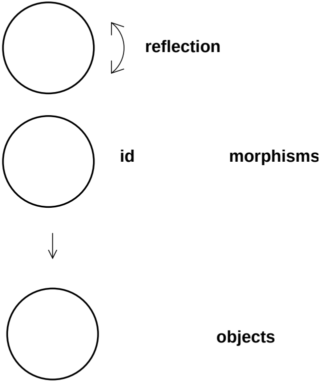
Manifolds and Smooth Groupoids

A manifold can be viewed as a smooth topological equivalence relation.

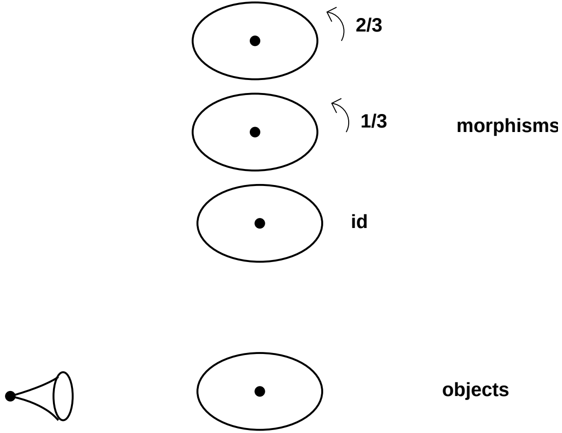


Orbifolds and Smooth Groupoids

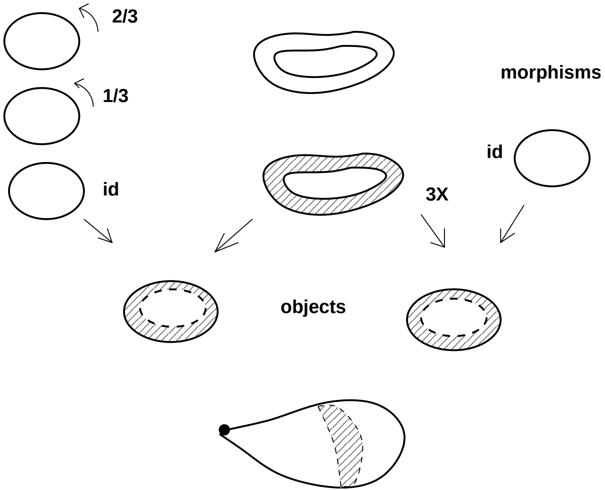
Example 1: a Global Quotient Orbifold Take $X = S^1$ with the $\mathbb{Z}/2$-action by reflection.



Example 2: a Single Chart Orbifold We model an order 3 cone with



Example 3: an orbifold atlas with several charts For the teardrop we obtain:



Smooth Groupoids

- A Lie groupoid $\mathcal{G}$ is a groupoid in the category of smooth manifolds

$$G_1 \times_{s,G_0,t} G_1 \;\substack{\xrightarrow{\pi_1} \\ \xrightarrow{m} \\ \xrightarrow[\pi_2]{}}\; G_1 \xrightarrow{i} G_1 \;\substack{\xrightarrow{s} \\ \xleftarrow{u} \\ \xrightarrow[t]{}}\; G_0$$

such that the source and target maps are submersions.

- Homomorphisms $\varphi\colon \mathcal{G} \to \mathcal{H}$ are internal functors,

$$\begin{array}{ccc} G_1 & \xrightarrow{\varphi_1} & H_1 \\ \downarrow\uparrow\downarrow & & \downarrow\uparrow\downarrow \\ G_0 & \xrightarrow[\varphi_0]{} & H_0 \end{array}$$

- 2-Cells $\alpha\colon \varphi \Rightarrow \psi$ are internal natural transformations,

$$\begin{array}{ccc} G_1 & \substack{\xrightarrow{\varphi_1} \\ \xrightarrow[\psi_1]{}} & H_1 \\ s \downarrow\uparrow\downarrow t & \nearrow \alpha & s \downarrow\uparrow\downarrow t \\ G_0 & \substack{\xrightarrow{\varphi_0} \\ \xrightarrow[\psi_0]{}} & H_0 \end{array}$$

Lie Groupoid Examples

- Manifolds 1: $G_0 = G_1 = M$ with all structure maps identities.
- Manifolds 2: G_0 is the disjoint union of charts and G_1 is the disjoint union of all the intersections of pairs of charts (with source and target maps the appropriate embeddings).
- Lie groups: $G_0 = \{\bullet\}$ and $G_1 = L$, a Lie group.
- Translation Groupoids: for a Lie group L acting on a manifold M, there is a translation groupoid $L \ltimes M$,

$$L \times L \times M \xrightarrow{\mu \times 1_M} L \times M \xrightarrow{(\iota, a)} L \times M \underset{\pi_2}{\overset{a}{\rightrightarrows}} M$$

Lie Groupoid Examples

- Manifolds 1: $G_0 = G_1 = M$ with all structure maps identities.
- Manifolds 2: G_0 is the disjoint union of charts and G_1 is the disjoint union of all the intersections of pairs of charts (with source and target maps the appropriate embeddings).
- Lie groups: $G_0 = \{\bullet\}$ and $G_1 = L$, a Lie group.
- Translation Groupoids: for a Lie group L acting on a manifold M, there is a translation groupoid $L \ltimes M$,

$$L \times L \times M \xrightarrow{\mu \times 1_M} L \times M \xrightarrow{(\iota, a)} L \times M \underset{\pi_2}{\overset{a}{\rightrightarrows}} M$$

Orbifold Groupoids

- For an orbifold atlas $\mathcal{U}$ there is an atlas groupoid $\mathcal{G}(\mathcal{U})$.
- Its manifold of objects is the disjoint union of charts.
- Its manifold of arrows is a quotient of the space

$$\coprod_{\lambda_1 \colon U \hookrightarrow V_1,\, \lambda_2 \colon U \hookrightarrow V_2} U$$

- Atlas groupoids are étale in the sense that the source and target maps are local diffeomorphisms.
- Atlas groupoids are proper in the sense that the map $(s, t) \colon G_1 \to G_0 \times G_0$ is proper.
- However, equivalent atlases do not necessarily give rise to isomorphic or equivalent groupoids.

Morita Equivalence for Groupoids

Theorem (Morita Equivalence for Groupoids in Set)

Given two small groupoids $\mathcal{G}$ and $\mathcal{H}$, the following are equivalent:

1. *there is an equivalence of categories* $\mathrm{Hom}(\mathcal{G}^{op}, \mathrm{Set}) \simeq \mathrm{Hom}(\mathcal{H}^{op}, \mathrm{Set})$*;*
2. $\mathcal{G}$ *and* $\mathcal{H}$ *are equivalent;*
3. *there are weak equivalences* $\mathcal{G} \longleftarrow \mathcal{K} \longrightarrow \mathcal{H}$*;*
4. *there are profunctors (or, bimodules)* $\varphi\colon \mathcal{G} \longrightarrow\!\!\!\!\!\!\!+\;\; \mathcal{H}$ *and* $\psi\colon \mathcal{H} \longrightarrow\!\!\!\!\!\!\!+\;\; \mathcal{G}$ *such that* $\psi \circ \varphi \cong 1_{\mathcal{G}}$ *and* $\varphi \circ \psi \cong 1_{\mathcal{H}}$.

Essential Equivalences of Smooth Groupoids

Definition

A morphism $\varphi\colon \mathcal{G} \to \mathcal{H}$ of Lie groupoids is an **essential equivalence** when it satisfies the following two conditions:

- φ is **essentially surjective on objects** in the sense that $t \circ \pi_2$ is a surjective submersion:

$$\begin{array}{ccccc} G_0 \times_{H_0} H_1 & \xrightarrow{\pi_2} & H_1 & \xrightarrow{t} & H_0 \\ {\scriptstyle \pi_1}\downarrow & & \downarrow{\scriptstyle s} & & \\ G_0 & \xrightarrow[\varphi_0]{} & H_0 & ; & \end{array}$$

- φ is **fully faithful** in the sense that the following diagram is a pullback:

$$\begin{array}{ccc} G_1 & \xrightarrow{\varphi_1} & H_1 \\ {\scriptstyle (s,t)}\downarrow & & \downarrow{\scriptstyle (s,t)} \\ G_0 \times G_0 & \xrightarrow{\varphi_0 \times \varphi_0} & H_0 \times H_0 \end{array}$$

Orbifold Groupoids

Theorem

For Lie groupoids $\mathcal{G}$ and $\mathcal{H}$ the following are equivalent:

- $\mathrm{Sh}(\mathcal{G}) \simeq \mathrm{Sh}(\mathcal{H})$;
- *There exists a groupoid $\mathcal{K}$ with essential equivalences*

$$\mathcal{G} \longleftarrow \mathcal{K} \longrightarrow \mathcal{H}.$$

Remarks

- Two such groupoids $\mathcal{G}$ and $\mathcal{H}$ are called Morita equivalent.
- Equivalent orbifold atlases give rise to Morita equivalent groupoids.
- An orbifold groupoid is a groupoid which is Morita equivalent to a proper étale groupoid.
- An orbifold is representable when its atlas groupoid is Morita equivalent to a translation groupoid $L \ltimes M$.

Orbifold Groupoids

Theorem

For Lie groupoids $\mathcal{G}$ and $\mathcal{H}$ the following are equivalent:

- $\mathrm{Sh}(\mathcal{G}) \simeq \mathrm{Sh}(\mathcal{H})$;
- *There exists a groupoid $\mathcal{K}$ with essential equivalences*

$$\mathcal{G} \longleftarrow \mathcal{K} \longrightarrow \mathcal{H}.$$

Remarks

- Two such groupoids $\mathcal{G}$ and $\mathcal{H}$ are called Morita equivalent.
- Equivalent orbifold atlases give rise to Morita equivalent groupoids.
- An orbifold groupoid is a groupoid which is Morita equivalent to a proper étale groupoid.
- An orbifold is representable when its atlas groupoid is Morita equivalent to a translation groupoid $L \ltimes M$.

Orbifold Groupoids

Theorem

For Lie groupoids $\mathcal{G}$ and $\mathcal{H}$ the following are equivalent:

- $\mathrm{Sh}(\mathcal{G}) \simeq \mathrm{Sh}(\mathcal{H})$;
- *There exists a groupoid $\mathcal{K}$ with essential equivalences*

$$\mathcal{G} \longleftarrow \mathcal{K} \longrightarrow \mathcal{H}.$$

Remarks

- Two such groupoids $\mathcal{G}$ and $\mathcal{H}$ are called Morita equivalent.
- Equivalent orbifold atlases give rise to Morita equivalent groupoids.
- An orbifold groupoid is a groupoid which is Morita equivalent to a proper étale groupoid.
- An orbifold is representable when its atlas groupoid is Morita equivalent to a translation groupoid $L \ltimes M$.

Orbifold Groupoids

Theorem

For Lie groupoids $\mathcal{G}$ and $\mathcal{H}$ the following are equivalent:

- $\mathrm{Sh}(\mathcal{G}) \simeq \mathrm{Sh}(\mathcal{H})$;
- *There exists a groupoid $\mathcal{K}$ with essential equivalences*

$$\mathcal{G} \longleftarrow \mathcal{K} \longrightarrow \mathcal{H}.$$

Remarks

- Two such groupoids $\mathcal{G}$ and $\mathcal{H}$ are called Morita equivalent.
- Equivalent orbifold atlases give rise to Morita equivalent groupoids.
- An orbifold groupoid is a groupoid which is Morita equivalent to a proper étale groupoid.
- An orbifold is representable when its atlas groupoid is Morita equivalent to a translation groupoid $L \ltimes M$.

The Bicategory of Orbifolds

Theorem

There is a bicategory of fractions **OrbiGrpd**$[W^{-1}]$ *of orbifolds where:*

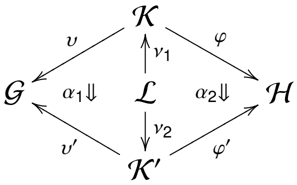
- **objects** *are orbifold groupoids;*
- **morphisms** *(generalized maps) are spans* $\mathcal{G} \stackrel{w}{\leftarrow} \mathcal{K} \stackrel{\varphi}{\rightarrow} \mathcal{H}$ *where w is an essential equivalence;*
- **2-cells** *are equivalence classes of diagrams of the form*

The Bicategory of Orbifolds

Theorem

There is a bicategory of fractions **OrbiGrpd**$[W^{-1}]$ *of orbifolds where:*

- **objects** *are orbifold groupoids;*
- **morphisms** *(generalized maps) are spans* $\mathcal{G} \xleftarrow{w} \mathcal{K} \xrightarrow{\varphi} \mathcal{H}$ *where* w *is an essential equivalence;*
- **2-cells** *are equivalence classes of diagrams of the form*



The Category of Translation Groupoids

- Let **rOrbiGrpd**$[W^{-1}]$ be the full sub-bicategory of representable orbifolds.
- Note:
  - Translation groupoid presentations for representable orbifolds are not unique (we still need to work up to Morita equivalence).
  - Generalized morphisms between translation groupoids may involve more general groupoids

$$G \ltimes M \longleftarrow \mathcal{K} \longrightarrow H \ltimes N$$

The Representable Orbifold Category

Define a bicategory of fractions $\mathbf{TrOrbiGrpd}[W^{-1}]$:

- **objects** are orbifold translation groupoids
- **morphisms** are spans

$$G \ltimes X \xleftarrow{(\upsilon,w)} K \ltimes Y \xrightarrow{(\varphi,f)} H \ltimes Z$$

  where (φ, f) is an equivariant map and (υ, w) is an equivariant essential equivalence

- **2-cells** are equivalence classes of diagrams of the form

$$\begin{array}{ccccc} & & K \ltimes Y & & \\ & \swarrow^{(\upsilon,w)} & \uparrow & \searrow^{(\varphi,f)} & \\ G \ltimes X & \alpha_1 \Downarrow & L \ltimes Y'' & \alpha_2 \Downarrow & H \ltimes Z \\ & \nwarrow_{(\upsilon',w')} & \downarrow & \nearrow_{(\varphi',f')} & \\ & & K' \ltimes Y' & & \end{array}$$

Equivalence of Bicategories

Theorem (P–Scull)

There is an equivalence of bicategories

$$\mathrm{rOrbiGrpd}\,[W^{-1}] \quad \simeq \quad \mathrm{TrOrbiGrpd}\,[W^{-1}]$$

representable orbifolds $\longleftrightarrow$ *(orbi) G-spaces*

Moreover, the equivariant essential equivalences can be factored into morphisms of the following forms:

- $G \ltimes X \to G/K \ltimes X/K$ for $K \trianglelefteq G$, where K acts freely on X;
- $L \ltimes Z \to H \ltimes (H \otimes_L Z)$ for $L \leq H$.

What have we learned so far?

- When we restrict ourselves to representable orbifolds, we only need to consider translation groupoids and (generalized) equivariant maps between them.
- Equivariant invariants are orbifold invariants iff they are invariant under the equivariant Morita equivalences
  - $G \ltimes X \to G/K \ltimes X/K$ for $K \trianglelefteq G$, where K acts freely on X;
  - $L \ltimes Z \to H \ltimes (H \otimes_L Z)$ for $L \leq H$.

Bredon Cohomology of G-Spaces

Let G be a Lie group or a topological group, and X a G-space.

Idea

Study the homotopy of X in terms of its diagram of fixed point subspaces

$$X^H := \{x;\ h \cdot x = x \text{ for all } h \in H\}$$

for all closed subgroups $H \leq G$, with arrows given by natural inclusions and the action of G.

This diagram is indexed by the orbit category $\mathcal{O}_G$.

Orbit Categories

Definition

The **orbit 2-category** $\mathcal{O}_G$ has

- **Objects**: G-sets G/H, for $H \leq G$;
- **Arrows**: G-maps $G/H \to G/K$.
- **Note**: for a G-space X, a G-map $\varphi\colon G/H \to X$ is determined by $x = \varphi(eH)$; moreover, $x \in X^H$.
- $\mathcal{O}_G(G/H, G/K) \cong (G/K)^H$ has the structure of a topological space.
- **2-Cells**: homotopy classes of paths.

Definition

The **homotopy orbit category** $h\mathcal{O}_G$ is the category of orbit types G/H, with homotopy classes (*i.e.*, connected components) of G-equivariant morphisms as arrows.

Orbit Categories

Definition

The **orbit 2-category** $\mathcal{O}_G$ has

- **Objects**: G-sets G/H, for $H \leq G$;
- **Arrows**: G-maps $G/H \to G/K$.
- **Note**: for a G-space X, a G-map $\varphi\colon G/H \to X$ is determined by $x = \varphi(eH)$; moreover, $x \in X^H$.
- $\mathcal{O}_G(G/H, G/K) \cong (G/K)^H$ has the structure of a topological space.
- **2-Cells**: homotopy classes of paths.

Definition

The **homotopy orbit category** $h\mathcal{O}_G$ is the category of orbit types G/H, with homotopy classes (*i.e.*, connected components) of G-equivariant morphisms as arrows.

Equivariant Coefficient Systems

- A G-space X gives rise to a functor

$$\Phi_X \colon \mathcal{O}_G^{op} \to \textbf{Spaces}, \quad \Phi_X(G/H) = \mathrm{Map}_G(G/H, X) = X^H.$$

- An **equivariant coefficient system** (with constant coefficients) is a functor

$$h\mathcal{O}_G^{op} \to \textbf{Ab}.$$

Equivariant Coefficient Systems

- A G-space X gives rise to a functor

$$\Phi_X\colon \mathcal{O}_G^{op} \to \mathbf{Spaces}, \quad \Phi_X(G/H) = \mathrm{Map}_G(G/H, X) = X^H.$$

- An **equivariant coefficient system** (with constant coefficients) is a functor

$$h\mathcal{O}_G^{op} \to \mathbf{Ab}.$$

Bredon Cohomology

- There is an equivariant chain complex:

$$\underline{C}_*(X)(G/H) = C_*(X^H/WH_0)$$

  where WH_0 is the identity component of the Weyl group NH/H.
- For each n, this gives rise to a coefficient system $\underline{C}_n(X)$.
- For any equivariant coefficient system $\underline{A}$, there is a cochain complex:

$$C^n_G(X; A) = \mathrm{Hom}_G(\underline{C}_n(X), \underline{A}).$$

- The Bredon cohomology of X is the cohomology of this cochain complex:

$$H^*_G(X; \underline{A}) = H^*_G(X; \underline{A}) = H^*(C^*_G(X; \underline{A}))$$

Bredon Cohomology

- There is an equivariant chain complex:

$$\underline{C}_*(X)(G/H) = C_*(X^H/WH_0)$$

  where WH_0 is the identity component of the Weyl group NH/H.
- For each n, this gives rise to a coefficient system $\underline{C}_n(X)$.
- For any equivariant coefficient system $\underline{A}$, there is a cochain complex:

$$C^n_G(X; A) = \mathrm{Hom}_G(\underline{C}_n(X), \underline{A}).$$

- The Bredon cohomology of X is the cohomology of this cochain complex:

$$H^*_G(X; \underline{A}) = H^*_G(X; \underline{A}) = H^*(C^*_G(X; \underline{A}))$$

Example

Let $G = \mathbb{Z}/2$ act on S^1 by reflection in the line connecting the north and south pole.

- The orbit category $\mathcal{O}_{\mathbb{Z}/2}$: $\sigma \circlearrowright G \longrightarrow 0$.
- Coefficient systems:

$$\tau \circlearrowright B \xleftarrow{\iota} A, \qquad \text{such that } \tau^2 = 1_B \text{ and } \tau\iota = \iota.$$

Examples:

1. $B = A = \mathbb{Z}$ and all structure maps are identities;
2. $B = \mathbb{Z} \oplus \mathbb{Z}$, $A = \mathbb{Z}$, τ is interchange and ι is the diagonal;
3. $B = 0$ and $A = \mathbb{Z}$.

The resulting cohomology groups are:

1. the cohomology of the orbitspace;
2. the cohomology of S^1;
3. the cohomology of the fixed point set.

Example

Let $G = \mathbb{Z}/2$ act on S^1 by reflection in the line connecting the north and south pole.

- The orbit category $\mathcal{O}_{\mathbb{Z}/2}$: $\sigma \circlearrowright G \longrightarrow 0$.
- Coefficient systems:

$$\tau \circlearrowright B \xleftarrow{\iota} A, \qquad \text{such that } \tau^2 = 1_B \text{ and } \tau\iota = \iota.$$

Examples:

1. $B = A = \mathbb{Z}$ and all structure maps are identities;
2. $B = \mathbb{Z} \oplus \mathbb{Z}$, $A = \mathbb{Z}$, τ is interchange and ι is the diagonal;
3. $B = 0$ and $A = \mathbb{Z}$.

The resulting cohomology groups are:

1. the cohomology of the orbitspace;
2. the cohomology of S^1;
3. the cohomology of the fixed point set.

Example

Let $G = \mathbb{Z}/2$ act on S^1 by reflection in the line connecting the north and south pole.

- The orbit category $\mathcal{O}_{\mathbb{Z}/2}$: $\sigma \circlearrowright G \longrightarrow 0$.
- Coefficient systems:

$\tau \circlearrowright B \xleftarrow{\iota} A$, such that $\tau^2 = 1_B$ and $\tau\iota = \iota$.

Examples:

1. $B = A = \mathbb{Z}$ and all structure maps are identities;
2. $B = \mathbb{Z} \oplus \mathbb{Z}$, $A = \mathbb{Z}$, τ is interchange and ι is the diagonal;
3. $B = 0$ and $A = \mathbb{Z}$.

The resulting cohomology groups are:

1. the cohomology of the orbitspace;
2. the cohomology of S^1;
3. the cohomology of the fixed point set.

Example

Let $G = \mathbb{Z}/2$ act on S^1 by reflection in the line connecting the north and south pole.

- The orbit category $\mathcal{O}_{\mathbb{Z}/2}$: $\sigma \circlearrowright G \longrightarrow 0$.
- Coefficient systems:

$$\tau \circlearrowright B \xleftarrow{\iota} A, \qquad \text{such that } \tau^2 = 1_B \text{ and } \tau\iota = \iota.$$

Examples:

1. $B = A = \mathbb{Z}$ and all structure maps are identities;
2. $B = \mathbb{Z} \oplus \mathbb{Z}$, $A = \mathbb{Z}$, τ is interchange and ι is the diagonal;
3. $B = 0$ and $A = \mathbb{Z}$.

The resulting cohomology groups are:

1. the cohomology of the orbitspace;
2. the cohomology of S^1;
3. the cohomology of the fixed point set.

Constant Orbifold Coefficients

Let $(\varphi, f)\colon G \ltimes X \to G' \ltimes X'$ be an equivariant map between orbifold groupoids.

- This gives rise to a homomorphism

$$H^*_{G'}(X'; \underline{A}) \to H^*_G(X; \varphi^*\underline{A}).$$

- [Moerdijk-Svensson] If $\varphi\colon G \to K$ is any group homomorphism, then

$$H^*_K(K \times_{\varphi,G} X; \underline{A}) \cong H^*_G(X; \varphi^*\underline{A})$$

  where $K \times_{\varphi,G} X = K \times X/(k, gx) \sim (k\varphi(g), x)$.
- An essential equivalence (φ, f) induces an isomorphism

$$H^*_{G'}(X'; \underline{A}) \cong H^*_G(X; \varphi^*\underline{A}).$$

Let $(\varphi, f)\colon G \ltimes X \to G' \ltimes X'$ be an essential equivalence of orbifolds.

- If $r_X\underline{A} = r_X\underline{B}$ then $H^*_G(X, \underline{A}) = H^*_G(X, \underline{B})$.
- For each orbifold coefficient system $\underline{B}$ on $\mathcal{O}_G$, there is an orbifold coefficient system $\underline{A}$ on $\mathcal{O}_{G'}$ such that

$$r_X\varphi^*\underline{A} = r_X\underline{B}.$$

- For each orbifold coefficient system $\underline{B}$ on $\mathcal{O}_G$, there is an orbifold coefficient system $\underline{A}$ on $\mathcal{O}_{G'}$ such that

$$H^*_G(X; \underline{B}) \cong H^*_{G'}(X'; \underline{A}).$$

The Fundamental Groupoid

Definition

For a discrete group G and a G-space X, the functor

$$\Pi_X : \mathcal{O}_G \to \mathbf{Cat}$$

is defined by

- $\Pi_X(G/H)$ is the category with
  - **Objects**: $x : G/H \to X$ (points in X^H);
  - **Arrows**: homotopy classes of paths $[\gamma] : G/H \times I \to X$.
- For $\alpha : G/H \to G/K$, $\Pi_X(\alpha)$ is defined by composition.

Definition

For a discrete group G and a G-space X, the fundamental groupoid is defined by the Grothendieck construction

$$\Pi_G(X) = \int_{\mathcal{O}_G} \Pi_X.$$

So it has

- **Objects**: $(G/H, x\colon G/H \to X)$
- **Arrows**:

$$(\alpha, [\gamma])\colon (G/H, x\colon G/H \to X) \to (G/K, y\colon G/K \to X)$$

  where $\alpha\colon G/H \to G/K$ in $\mathcal{O}_G$ and $[\gamma]\colon x \to y \circ \alpha$ is a homotopy class of paths in X^H.

The Orbifold Fundamental Groupoid

Definition

For an orbifold $G \ltimes X$, the 2-functor $\Pi_X \colon \mathcal{O}_G \to \textbf{Cat}$ is defined by

- $\Pi_X(G/H)$ is the category with
  - Objects: $\varphi \colon G/H \to X$ (points in X^H);
  - Arrows: homotopy classes of paths $[\gamma] \colon G/H \times I \to X$.
- For $\alpha \colon G/H \to G/K$, $\Pi_X(\alpha)$ is defined by composition with α.
- For a homotopy class ξ of paths from α to α' in $\mathcal{O}_G[G/H, G/K]$,

  $$\Pi_X(\xi) \colon \Pi_X(\alpha) \to \Pi_X(\alpha')$$

  is the natural transformation with components

  $$\Pi_X(\xi)_x \colon I \to X^H, \qquad \Pi_X(\xi)_x(s) = x \circ \xi(s)$$

  for $x \colon G/K \to X$.

Definition

For an orbifold $G \ltimes X$, the orbifold fundamental groupoid is $\Pi_G(X) = \int_{\mathcal{O}_G} \Pi_X$, with

- **Objects**: $(G/H, \varphi\colon G/H \to X)$, or $(G/H, x \in X^H)$;
- **Arrows**: $(\alpha, \gamma)\colon (G/H, x \in X^H) \to (G/K, y \in X^K)$ where $\alpha\colon G/H \to G/K$ in $\mathcal{O}_G$ and $[\gamma]\colon x \to y \circ \alpha$ is a homotopy class of paths in X^H.
- **2-Cells**: a 2-cell $(\alpha, \gamma) \Rightarrow (\alpha', \gamma')$ is determined by a homotopy class of paths $[\xi]\colon \alpha \to \alpha'$ in $\mathcal{O}_G[G/H, G/K]$ such that the following diagram commutes

$$\begin{array}{ccc} x & \xrightarrow{[\gamma]} & y \circ \alpha \\ {\scriptstyle x}\Big| & & \Big|{\scriptstyle y \circ [\xi]} \\ x & \xrightarrow[{[\gamma']}]{} & y \circ \alpha' \end{array} \quad .$$

Definition

The discrete orbifold fundamental groupoid $\Pi_G^d(X)$ is obtained from $\Pi_G(X)$ by taking isomorphism classes of arrows.

Theorem

The Π construction induces a 2-functor

$$\mathbf{TrOrbiGrpd} \to \mathbf{2\text{-}Cat}.$$

Moreover, this functor sends essential equivalences to weak equivalences of 2-categories. Consequently, Π^d induces a 2-functor

$$\mathbf{TrOrbiGrpd} \to \mathbf{Cat}$$

which sends essential equivalence to weak equivalences of categories.

More General Coefficients

For discrete groups, Moerdijk and Svensson generalized the Bredon cohomology theory in terms of cohomology of small categories and extended the coefficients as follows.

- They construct a diagram

$$\begin{array}{ccc} \Delta_G(X) & \xrightarrow{\;u_X\;} & \Pi_G(X) \\ & {\scriptstyle p_X} \searrow \quad \swarrow {\scriptstyle q_X} & \\ & \mathcal{O}_G & \end{array}$$

- A set of general coefficients is a functor $\underline{A}\colon \Delta_G(X)^{\mathrm{op}} \to \mathbf{Ab}.$
- Twisted coefficients are given by a functor $\underline{M}\colon \Pi_G(X)^{\mathrm{op}} \to \mathbf{Ab}.$
- The cohomology with twisted coefficients is calculated as the cohomology of the nerve of $\Delta_G(X)$ with the corresponding coefficients,

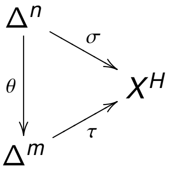
$$H^*_G(X; \underline{M}) = H^*(\Delta_G(X); \underline{M} \circ u_X).$$

The Singular Representation Category

Definition

For a discrete group G and a G-space X, the functor $\Delta_X\colon \mathcal{O}_G \to \mathbf{Cat}$ is defined by

- $\Delta_X(G/H)$ is the category with objects $\sigma\colon \Delta^n \to X^H$ and arrows simplicial operators



- For $\alpha\colon G/H \to G/K$, $\Delta_X(\alpha)$ is defined by composition with α.

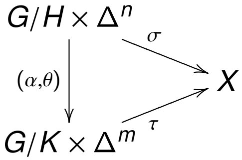
Definition

The **singular representation category** is defined by the Grothendieck construction $\Delta_G(X) = \int_{\mathcal{O}_G} \Delta_X$. So it has

- **Objects**: $(G/H, \sigma\colon G/H \times \Delta^n \to X)$
- **Arrows**:

$$(\alpha, \theta)\colon (G/H, \sigma\colon G/H \times \Delta^n \to X) \to (G/K, \tau\colon G/K \times \Delta^m \to X)$$

where $\alpha\colon G/H \to G/K$ in $\mathcal{O}_G$ and $\theta\colon \Delta^n \to \Delta^m$ such that

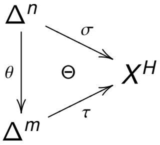


The Orbifold Singular Representation

Definition

For an orbifold $G \ltimes X$, define the 2-functor $\Delta_X \colon \mathcal{O}_G \to \textbf{Cat}$ by

- $\Delta_X(G/H)$ is the category with
  - Objects $\sigma \colon \Delta^n \to X^H$, i.e., $\sigma \colon G/H \times \Delta^n \to X$.
  - Arrows



where θ is a simplicial operator and $\Theta \colon \Delta^n \times I \to X^H$ a homotopy class of homotopies in X^H.

- For $\alpha \colon G/H \to G/K$, $\Delta_X(\alpha)$ is by composition with α.
- For a 2-cell $\xi \colon \alpha \Rightarrow \alpha'$, $\Delta_X(\xi)$ has components $\Delta_X(\xi)_\sigma = (\theta_\xi, \Theta_\xi) \colon \sigma \circ (\alpha \times \Delta^n) \to \sigma \circ (\alpha' \times \Delta^n)$ with $\theta_\xi = 1_{\Delta^n}$, and $\Theta_\xi \colon G/H \times \Delta^n \times I \to X$, defined by $\Theta_\xi(-,-,s) = \sigma \circ (\xi(s) \times \Delta^n)$.

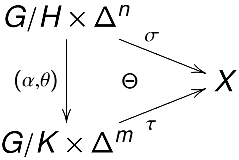
Definition

The orbifold singular representation category is the 2-category $\Delta_G(X) = \int_{\mathcal{O}_G} \Delta_X$, with

- **Objects**: $(G/H, \sigma\colon G/H \times \Delta^n \to X)$
- **Arrows**:

$$(\alpha, \theta, \Theta)\colon (G/H, \sigma\colon G/H \times \Delta^n \to X) \to (G/K, \tau\colon G/K \times \Delta^m \to X)$$

where $\alpha\colon G/H \to G/K$ in $\mathcal{O}_G$, $\theta\colon \Delta^n \to \Delta^m$, and $\Theta\colon G/H \times \Delta^n \times I \to X$ is a homotopy class of homotopies as in



- **2-Cells**: a cell $[\xi]\colon (\alpha_1, \theta_1, \Theta_1) \to (\alpha_2, \theta_2, \Theta_2)$ is a homotopy class of paths $[\xi]\colon \alpha_1 \to \alpha_2$ such that

$$\begin{array}{ccc} \sigma & \xrightarrow{(\theta_1,[\Theta_1])} & \Delta_X(\alpha_1)(\tau) \\ \| & & \downarrow{\scriptstyle \Delta_X(\xi)_\tau=(\theta_\xi,[\Theta_\xi])} \\ \sigma & \xrightarrow[(\theta_2,[\Theta_2])]{} & \Delta_X(\alpha_2)(\tau). \end{array}$$

Definition

The discrete orbifold singular representation category $\Delta_G^d(X)$ is the category obtained from $\Delta_G(X)$ by quotienting out the 2-cells.

Morita Equivalences

Theorem

The Δ construction induces a 2-functor **TrOrbiGrpd** $\to$ **2-Cat***. Moreover this functor sends essential equivalences to weak equivalences of 2-categories.*

Theorem

The Δ^d construction induces a 2-functor from **TrOrbiGrpd** $\to$ **Cat***. Moreover this functors sends essential equivalences to weak equivalences of categories.*

$$\begin{array}{ccccc}
\Delta_G(X) & \xrightarrow{u_X} & \Pi_G(X) & & \\
\downarrow \sim & & \downarrow \sim & \searrow^{\underline{M}} & \\
\Delta_H(Y) & \xrightarrow{u_Y} & \Pi_H(Y) & \xrightarrow{\underline{N}} & \mathbf{Ab}
\end{array}$$