Applications of the Weighted Limit Theorem to Hopf Algebra Theory

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Outline

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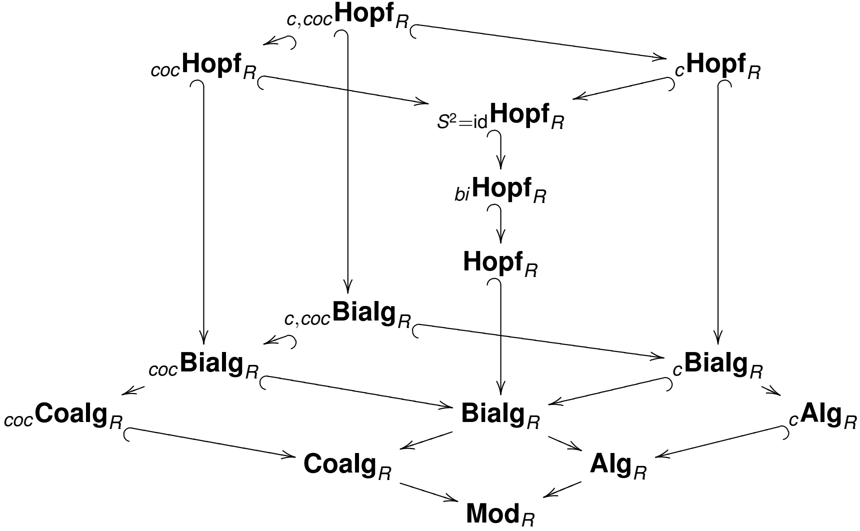
Agenda

What can be said about the following (categories and) functors?



Recall

1. $\mathbf{Bialg}_R = \mathbf{Mon}(\mathbf{Coalg}_R) = \mathbf{Comon}(\mathbf{Alg}_R)$.
2. $\mathbf{Hopf}_R$, the category of *Hopf algebras* is the full subcategory of $\mathbf{Bialg}_R$ spanned by those bialgebras (B, m, e, μ, ϵ) which carry an *antipode*, i.e., an R-linear map $S\colon B \to B$ such that

$$B \xrightarrow{\mu} B \otimes B \xrightarrow{S \otimes \mathrm{id}_B} B \otimes B \to B \;=\; B \xrightarrow{\epsilon} I \xrightarrow{e} B$$
$$B \xrightarrow{\mu} B \otimes B \xrightarrow{\mathrm{id}_B \otimes S} B \otimes B \to B \;=\; B \xrightarrow{\epsilon} I \xrightarrow{e} B$$

Then

1. The antipode is a bialgebra homomorphism $B \to B^{\mathrm{op},cop}$.
2. Every bialgebra homomorphism between Hopf monoids commutes with the antipodes.

Note

- $\mathbf{Hopf}_R$ is a full subcategory of the category of functor algebras $\mathbf{Alg}(-)^{\mathrm{op,cop}}$.
- $(\mathbf{Hopf}_R)^{\mathrm{op}} = \mathbf{Hopf}(\mathbf{Mod}_R^{\mathrm{op}})$

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Some subcategories

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- ${}_{coc}\mathbf{Hopf}_R = \mathbf{Comon}(\mathbf{Hopf}_R) = \mathbf{Hopf}(\mathbf{Coalg}_R)$
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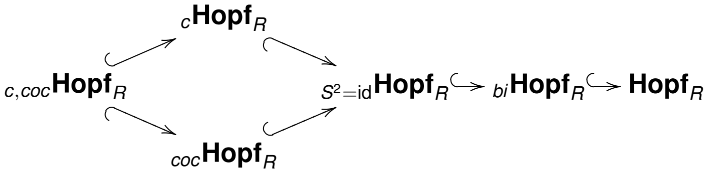

History (< 2008)

1. $\mathbf{Coalg}_k \to \mathbf{Vect}_k$ has a right adjoint (traditional "quasi-construction")
2. $\mathbf{Coalg}_R \to \mathbf{Mod}_R$ has a right adjoint for any commutative ring R [Barr, 1976] (by use of SAFT)
3. $\mathbf{Hopf}_k \to \mathbf{Coalg}_k$ has a left adjoint [Takeuchi, 1972] (in fact essentially a construction of a reflection $\mathbf{Bialg}_k \to \mathbf{Hopf}_k$)
4. ${}_{bi}\mathbf{Hopf}_k$ is reflective in $\mathbf{Hopf}_k$ [Schauenburg, 2000] (by construction)

Here: Follow Barr's approach.

Agenda

The functors **Mon** and **Comon**

Every monoidal functor $F\colon \mathbb{C} \to \mathbb{C}'$ induces functors $\hat{F}\colon \mathbf{Mon}\mathbb{C} \to \mathbf{Mon}\mathbb{C}'$ and (dually) $F^*\colon \mathbf{Comon}\mathbb{C} \to \mathbf{Comon}\mathbb{C}'$ such that

$$\begin{array}{ccc} \mathbf{Mon}\mathbb{C} & \xrightarrow{\hat{F}} & \mathbf{Mon}\mathbb{C}' \\ {\scriptstyle |-|}\downarrow & & \downarrow{\scriptstyle |-|} \\ \mathbb{C} & \xrightarrow{F} & \mathbb{C}' \end{array} \qquad \begin{array}{ccc} \mathbf{Comon}\mathbb{C} & \xrightarrow{F^*} & \mathbf{Comon}\mathbb{C}' \\ {\scriptstyle |-|}\downarrow & & \downarrow{\scriptstyle |-|} \\ \mathbb{C} & \xrightarrow{F} & \mathbb{C}' \end{array}$$

F has a right adjoint $\implies$ $\hat{F}$ has a right adjoint and, dually,
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These adjoints commute with the underlying functors as well.

By the Eckmann-Hilton argument this reduces the problem of finding adjunctions in the previous diagram as follows:

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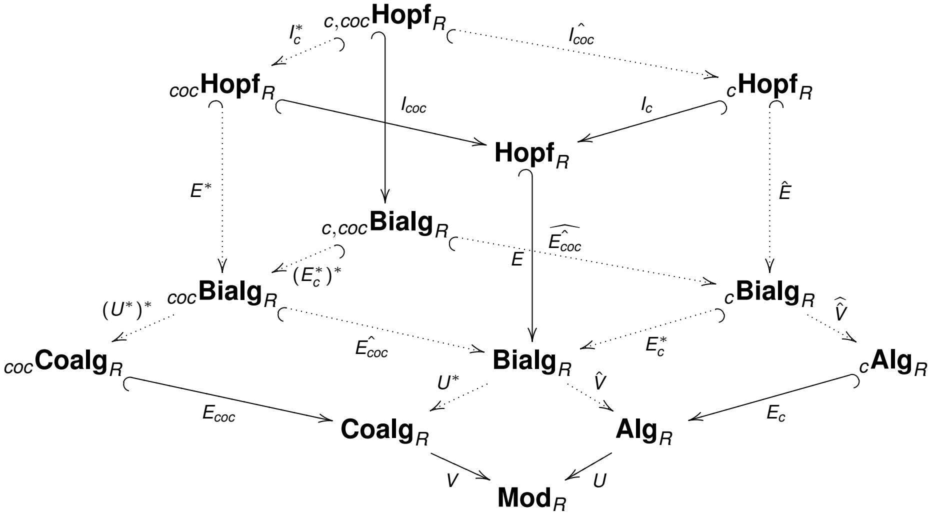
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I_c^*
$I_{coc}^{\hat{}}$
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E
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(Co)Monoids as Functor (Co)Algebras

1. Consider ${}_c\mathbf{Mon}\mathbb{C} \hookrightarrow \mathbf{Mon}\mathbb{C} \hookrightarrow \mathbf{Alg}(\otimes^2 + I) \xrightarrow{U} \mathbb{C}$.
   - If $\mathbb{C}$ has limits, then U creates limits (and directed colimits) and ${}_c\mathbf{Mon}\mathbb{C}$ and $\mathbf{Mon}\mathbb{C}$ are closed under those.
   - If $\mathbb{C}$ has (strong epi, mono)-factorizations and $\otimes^2$ preserves strong epis, then U creates (strong epi, mono)-factorizations and ${}_c\mathbf{Mon}\mathbb{C}$ and $\mathbf{Mon}\mathbb{C}$ are closed under these factorizations.
2. Dually, for ${}_{coc}\mathbf{Comon}\mathbb{C} \hookrightarrow \mathbf{Comon}\mathbb{C} \hookrightarrow \mathbf{Coalg}(\otimes^2 \times I) \xrightarrow{V} \mathbb{C}$:
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In $\mathbf{Mod}_R$ the functor $\otimes^2$ preserves strong epis and directed colimits for every R, and strong monos for every (von Neumann) regular R.

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A (co)limit construction

The following is a known generalization of the free group construction:

Assume that $\mathbf{X}$ has colimits and (strong epi, mono)-factorizations and is strong epi-cowellpowered. Let $U\colon \mathbf{A} \to \mathbf{X}$ be a faithful functor, which has a left adjoint and creates (strong epi, mono)-factorizations (call then *U strongly algebraic*).

Then a diagram D in $\mathbf{A}$ has a colimit $D_i \xrightarrow{\kappa_i} D$ with

$$U\kappa_i = UD_i \xrightarrow{\lambda_i} X \xrightarrow{\eta} UX^\sharp \xrightarrow{Ue} UD$$

where

1. $UD_i \xrightarrow{\lambda_i} X$ is a colimit of UD in $\mathbf{X}$,
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This construction not only applies to $\mathbf{Alg}_R \xrightarrow{U} \mathbf{Mod}_R$, but also (dually) to the construction of limits along $\mathbf{Coalg}_R \xrightarrow{V} \mathbf{Mod}_R$ provided that V has a right adjoint and R is regular.

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WLT

1. For any accessible functor $F\colon \mathbf{C} \to \mathbf{C}$ the categories $\mathbf{Alg}F$ and $\mathbf{Coalg}F$ are accessible.
2. For any pair of accessible functors $F, G\colon \mathbf{C} \to \mathbf{C}$ and any pair of natural transformations $\phi, \psi\colon F \Rightarrow G$ the equifier of (ϕ, ψ) is accessible.

$\mathbf{Alg}_R$ and $\mathbf{Coalg}_R$ can be described as equifiers in $\mathbf{Alg}(\otimes^2 + R)$ and $\mathbf{Coalg}(\otimes^2 \times R)$ respectively. All functors involved preserve directed colimits since $C \otimes -$ and $\otimes^2$ do.

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Alg$_R$ and **Coalg**$_R$

Proposition

1. **Alg**$_R$ *is locally presentable;*
2. $U \colon \mathbf{Alg}_R \to \mathbf{Mod}_R$ *is strongly algebraic;*
3. $_c\mathbf{Alg}_R$ *is locally presentable and reflective in* **Alg**$_R$.

Proposition

1. **Coalg**$_R$ *is locally presentable.*
2. $V \colon \mathbf{Coalg}_R \to \mathbf{Mod}_R$ *has a right adjoint; V is strongly co-algebraic, if R is regular.*
3. $_{coc}\mathbf{Coalg}_R$ *is locally presentable and coreflective in* **Coalg**$_R$.

$\mathbf{Coalg}_R$

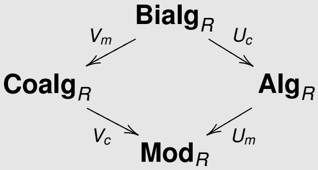
- $\mathbf{Coalg}(\otimes^2 \times R)$ is accessible by the WLT since $\otimes^2$ preserves directed colimits.
- $\mathbf{Coalg}_R$ and ${}_{coc}\mathbf{Coalg}_R$ are (again by the WLT) accessible as equifiers of pairs of natural transformations between finitary functors.
- All categories thus are locally presentable, since they are cocomplete.
- ${}_{coc}\mathbf{Coalg}_R \hookrightarrow \mathbf{Coalg}_R$ and $\mathbf{Coalg}_R \xrightarrow{V} \mathbf{Mod}_R$ now have right adjoints by the SAFT.

Agenda

Bialg$_R$

Proposition

In the diagram



1. **Bialg**$_R$ *is locally presentable,*
2. V_m *is monadic and* U_c *is comonadic,*
3. U_m *is strongly algebraic;* V_c *is strongly co-algebraic if R is regular.*

- The adjoints of U_m and V_c lift to those of V_m and U_c.
- V_m is finitary monadic; hence, since **Coalg**$_R$ is locally presentable, so is **Bialg**$_R$.

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(Co)Limits

Proposition

$\mathbf{Hopf}_R$ *is closed in* $\mathbf{Bialg}_R$ *under colimits. It is closed under limits as well, provided that R is regular.*

Since $V\colon \mathbf{Bialg}_R \to \mathbf{Alg}_R$ is comonadic, a colimit of a diagram D in $\mathbf{Bialg}_R$ is given by a colimit $(VD_i \xrightarrow{\kappa_i} A)$ supplied with the unique (comultiplication) μ and the unique (counit) ϵ making the following diagrams commute

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$$\begin{array}{ccc} D_i & \xrightarrow{\kappa_i} & A \\ {\scriptstyle \mu_i}\downarrow & & \downarrow{\scriptstyle \mu} \\ D_i \otimes D_i & \xrightarrow[\kappa_i \otimes \kappa_i]{} & A \otimes A \end{array} \qquad \begin{array}{ccc} D_i & \xrightarrow{\kappa_i} & A \\ & {\scriptstyle \epsilon_i}\searrow & \downarrow{\scriptstyle \epsilon} \\ & & R \end{array}$$

If each D_i is a Hopf algebra with antipode S_i, the colimit property induces a morphism $S\colon A \to A^{\mathrm{op,cop}}$ such that

$$\begin{array}{ccc} A & \overset{S}{\dashrightarrow} & A^{\mathrm{op,cop}} \\ {\scriptstyle \kappa_i}\uparrow & & \uparrow{\scriptstyle \kappa_i} \\ D_i & \underset{S_i}{\longrightarrow} & D_i^{\mathrm{op,cop}} \end{array}$$

commutes.

Omitting the underlying functors, each colimit map κ_i is, by the discussion above, the composition

$$\kappa_i = D_i \overset{\lambda_i}{\to} C \overset{u}{\to} TC \overset{q}{\to} A$$

where $(C, (\lambda_i))$ is a colimit of D in **Mod**$_R$, u is the universal morphism from C into the free algebra TC over C, and q is an extremal epimorphism in **Alg**$_R$.

Back to 2
Back to bi

If each D_i is a Hopf algebra with antipode S_i, the colimit property induces a morphism $S\colon A \to A^{\mathrm{op,cop}}$ such that

$$\begin{array}{ccc} A & \overset{S}{\dashrightarrow} & A^{\mathrm{op,cop}} \\ {\scriptstyle \kappa_i}\uparrow & & \uparrow{\scriptstyle \kappa_i} \\ D_i & \underset{S_i}{\longrightarrow} & D_i^{\mathrm{op,cop}} \end{array}$$

commutes.
Omitting the underlying functors, each colimit map κ_i is, by the discussion above, the composition

$$\kappa_i = D_i \overset{\lambda_i}{\to} C \overset{u}{\to} TC \overset{q}{\to} A$$

where $\big(C, (\lambda_i)\big)$ is a colimit of D in $\mathbf{Mod}_R$, u is the universal morphism from C into the free algebra TC over C, and q is an extremal epimorphism in $\mathbf{Alg}_R$.

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Back to bi

Then, by a simple diagram chase, Diagram

$$e \circ \epsilon \circ (q \circ u) = m \circ (\mathrm{id} \otimes S) \circ \mu \circ (q \circ u).$$

Now one has

For any bialgebra homomorphism $S\colon B \to B^{\mathrm{op,cop}}$ the equalizer of $m \circ (\mathrm{id} \otimes S) \circ \mu$ and $e \circ \epsilon$ (in $\mathbf{Mod}_R$) is a subalgebra of B.

Thus,

$$e \circ \epsilon = m \circ (\mathrm{id} \otimes S) \circ \mu$$

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Note that this argument can be dualized, if R is regular, which proves the second statement.

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$\mathbf{Hopf}_R$ is locally presentable

Proposition

$\mathbf{Hopf}_R$ *is a locally presentable category.*

$\mathbf{Hopf}_R$ is cocomplete by the above and an equifier in the category of functor algebras $\mathbf{Alg}(-)^{\mathrm{op,cop}}$, which is accessible by the WLT.

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Free and cofree Hopf algebras

Proposition

1. *For every ring R* $\mathbf{Hopf}_R$ *is coreflective in* $\mathbf{Bialg}_R$ *and* $\mathbf{Hopf}_R \to \mathbf{Alg}_R$ *has a right adjoint.*
2. *For every regular ring R* $\mathbf{Hopf}_R$ *is reflective in* $\mathbf{Bialg}_R$ *and* $\mathbf{Hopf}_R \to \mathbf{Coalg}_R$ *has a left adjoint.*

For 1. use the SAFT.
For 2. use the fact, that a full subcategory of a locally λ-presentable category, closed under limits and (λ-directed) colimits, is locally presentable.

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Constructions

For (von Neumann) regular rings a constructive argument could be given: In

$$\mathbf{Hopf}_R \hookrightarrow \mathbf{Alg}(-)^{op,cop} \xrightarrow{W} \mathbf{Coalg}_R$$

- the embedding has a left adjoint, since $\mathbf{Hopf}_R$ is closed in $\mathbf{Alg}(-)^{op,cop}$ under products and (R is regular!) image factorizations ...
- W has a left adjoint since $(-)^{op,cop}$ preserves coproducts ...

These arguments give Takeuchi's construction and — obviously — can be dualized to provide a coreflection.

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Agenda

Proposition

For any ring R in the chain of subcategories

$$ {}_{coc}\mathbf{Hopf}_R \subset {}_{S^2=\mathrm{id}}\mathbf{Hopf}_R \subset \mathbf{Hopf}_R $$

every category is locally presentable and coreflective in each of its successors.
Thus, ${}_{coc}\mathbf{Hopf}_R$ is coreflective in ${}_{coc}\mathbf{Bialg}_R$ and ${}_{c,coc}\mathbf{Hopf}_R$ is locally presentable and coreflective in ${}_{c}\mathbf{Hopf}_R$.

${}_{S^2=\mathrm{id}}\mathbf{Hopf}_R$ is closed under colimits in $\mathbf{Hopf}_R$, as follows from the definition of the antipode on a colimit of Hopf algebras. ▸ Colimits
Closure of ${}_{coc}\mathbf{Hopf}_R$ under colimits is obvious. Coreflectivity follows by SAFT.
For local presentability use representations of ${}_{c,coc}\mathbf{Hopf}_R$, ${}_{coc}\mathbf{Hopf}_R$ and ${}_{S^2=\mathrm{id}}\mathbf{Hopf}_R$ as equifiers.

Proposition

For any regular ring R in the chain of subcategories

$$ {}_{c}\mathbf{Hopf}_R \subset {}_{S^2=\mathrm{id}}\mathbf{Hopf}_R \subset {}_{bi}\mathbf{Hopf}_R \subset \mathbf{Hopf}_R $$

every category reflective in each of its successors.
In addition, in this case the subcategories

$$ {}_{S^2=\mathrm{id}}\mathbf{Hopf}_R \subset {}_{bi}\mathbf{Hopf}_R \subset \mathbf{Hopf}_R $$

are coreflective and ${}_{bi}\mathbf{Hopf}_R$ *is locally presentable.*
Moreover, ${}_{c}\mathbf{Hopf}_R$ *is locally presentable and reflective in* ${}_{c}\mathbf{Bialg}_R$ *and* ${}_{c,coc}\mathbf{Hopf}_R$ *is reflective in* ${}_{coc}\mathbf{Hopf}_R$.

Proof

The inverse of a bijective antipode is a bialgebra (anti)isomorphism. Thus, if (co)limits of Hopf algebras are constructed as in $\mathbf{Bialg}_R$, and D is a digram in $\mathbf{Hopf}_R$ with bijective antipode for each D_i, then the family of inverses of these antipodes induces a morphism on the (co)limit which is an invers of the (co)limit's antipode by that's very construction ▸ Colimits and ${}_{bi}\mathbf{Hopf}_R$ is closed in $\mathbf{Hopf}_R$ w.r.t. (co)limits.

Essentially the same argument shows the closure of ${}_{S^2=\mathrm{id}}\mathbf{Hopf}_R$. Now use the fact that every subcategory of a locally presentable category closed under limits and colimits is reflective and locally presentable. The coreflecitivity statements then follow by SAFT.

The additional statements follow by representing ${}_c\mathbf{Hopf}_R$ as an equifier and the lifting of adjunctions for monoidal functors to categories of monoids.

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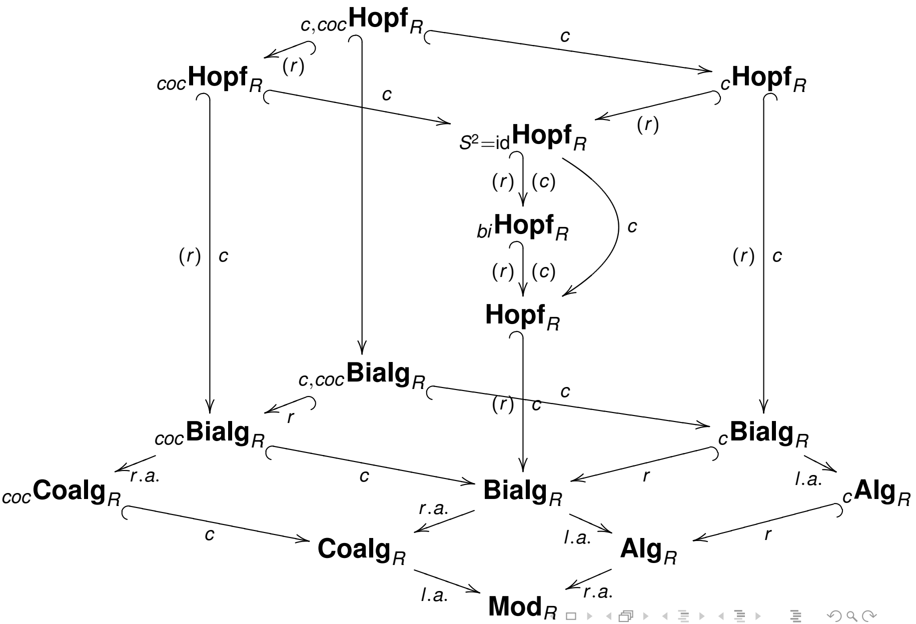
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Final remarks

The arguments used imply statements like

> *The cofree Hopf algebra on a commutative algebra is commutative* and *the free Hopf algebra on a cocommutative coalgebra is cocommutative.*

For regular rings R everything could be done without WLT. One even can show in this case that all categories considered are locally $\aleph_1$ (for $R = k$ a field even $\aleph_0$) presentable.

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Thanks for your attention!

$$\kappa_i \quad = \quad \begin{array}{ccc}
D_i \xrightarrow{\lambda_i} C \xrightarrow{u} TC \xrightarrow{q} A \\
\mu_i \downarrow \qquad\qquad\qquad\qquad \downarrow \mu \\
D_i \otimes D_i \xrightarrow{\kappa_i \otimes \kappa_i} A \otimes A \\
\mathrm{id} \otimes S_i \downarrow \qquad\qquad\qquad\qquad \downarrow \mathrm{id} \otimes S \\
D_i \otimes D_i \xrightarrow{\kappa_i \otimes \kappa_i} A \otimes A \\
m_i \downarrow \qquad\qquad\qquad\qquad \downarrow m \\
D_i \xrightarrow[\lambda_i]{} C \xrightarrow[u]{} TC \xrightarrow[q]{} A
\end{array}$$

$$\kappa_i \quad = \quad \begin{array}{ccc}
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Back