On Some Laws Relating Left and Right Actions

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ct2010 Genova

1. A complementation law relates left and right actions of categories

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2. Topological aspects of categories:
Temporal doctrines

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2. Topological aspects of categories: Temporal doctrines
3. Left actions act on right actions: Mixed doctrines

A category X acts on sets

Left actions are presheaves $A : X^{\mathrm{op}} \to \mathrm{Set}$.
Right actions are (covariant) presheaves $M : X \to \mathrm{Set}$.

$$X(x,y) \times Ay \to Ax \qquad ; \qquad Mx \times X(x,y) \to My$$

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A poset X acts on truth values $\{\mathtt{true}, \mathtt{false}\}$

Left actions are lower subsets A of X.

Right actions are upper subsets M of X.

$$x \leq y \, , \, y \in A \;\Rightarrow\; x \in A \qquad ; \qquad x \in M \, , \, x \leq y \;\Rightarrow\; y \in M$$

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A topological space X "acts" on truth values

Left actions are open subsets A of X.

Right actions are closed subsets M of X.

$$\xi \rightsquigarrow y \, , \, y \in A \;\Rightarrow\; \xi \in A \qquad ; \qquad \xi \in M \, , \, \xi \rightsquigarrow y \;\Rightarrow\; y \in M$$

Left or right actions organize themselves in indexed categories with good properties

Left or right actions of categories or of posets

$$\mathrm{Set}^{X^{\mathrm{op}}},\ X \in \mathrm{Cat} \qquad ; \qquad \mathrm{Set}^{X},\ X \in \mathrm{Cat}$$

$$\mathcal{D}X\,,\ X \in \mathrm{Pos} \qquad ; \qquad \mathcal{U}X\,,\ X \in \mathrm{Pos}$$

form a hyperdoctrine [Lawvere, 1969/1970] :

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form a hyperdoctrine [Lawvere, 1969/1970] :
a cartesian closed indexed category
with quantifications (the Kan extensions)
left and right adjoints to substitutions:

$$\exists_f \dashv f\cdot \dashv \forall_f$$

This is only partially true for actions of topological spaces.

Main question:
how can the left and right actions
be reciprocally related?

**How can be related
the left and the right actions of a category X?**

Rather tautological answer:
The left actions $A : X^{\mathrm{op}} \to \mathrm{Set}$ of a category X
are the right actions of the dual category X^{op},
and conversely.

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and conversely.
This approach does not show their interactions
and of course does not work for topology.

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How can be related the open and the closed parts of a topological space X?

Possible answer: complementation in $\mathcal{P}X$
gives a duality between open and closed parts:

$$\complement_X : \mathcal{O}X \cong (\mathcal{C}X)^{\mathrm{op}}$$

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How can be related
the lower and the upper parts of a poset X?

In this case, both the phenomena do appear:

$$\mathcal{D}X \cong \mathcal{U}X^{\mathrm{op}} \qquad ; \qquad \complement_X : \mathcal{D}X \cong (\mathcal{U}X)^{\mathrm{op}}$$

Our answer: Relative complementation

Left and right actions can be related by their inclusion
in a common ampler category $\mathcal{P}X$
and by the fact that exponentials in $\mathcal{P}X$ obey

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the complement of A relative to M,
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Indeed, for topological spaces (and in particular for posets) we rather trivially have:

If A is open and M is closed the complement of A relative to M, that is the exponential $A \Rightarrow M = \complement A \cup M$ in $\mathcal{P}X$ is closed (and conversely).

What is the common container for actions of categories?

The Grothendieck construction

Left and right actions of $X \in \mathrm{Cat}$
are fully included in categories over X,
as discrete fibrations and opfibrations:

$$\mathrm{Set}^{X^{\mathrm{op}}} \to \mathrm{Cat}/X \leftarrow \mathrm{Set}^{X}$$

The complementation law for categories

If A is a discrete fibration and M is a discrete opfibration
the complement of A relative to M,
that is the exponential $A \Rightarrow M$ in Cat/X
is a discrete opfibration (and conversely).

Interior and closure of a part of a topological space

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a key role is played by the interior and closure operators:

For a topological space X, we have
the interior coreflection and the closure reflection:

$$\Box_X : \mathcal{P}X \to \mathcal{O}X \qquad ; \qquad \Diamond'_X : \mathcal{P}X \to \mathcal{C}X$$

right and left adjoint to the inclusion
of open and closed parts in all subsets of X.

$$i_X : \mathcal{O}X \to \mathcal{P}X \qquad ; \qquad i'_X : \mathcal{C}X \to \mathcal{P}X$$

$$i_X \dashv \Box_X \qquad ; \qquad \Diamond'_X \dashv i'_X$$

Left and right interior and closure of a category over X

Given a category X, let

$$i_X : \mathrm{Set}^{X^{\mathrm{op}}} \to \mathrm{Cat}/X \qquad ; \qquad i'_X : \mathrm{Set}^{X} \to \mathrm{Cat}/X$$

be the inclusions of left and right action as discrete (op)fibrations over X.

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Theorem

These inclusions both have a reflection and a coreflection:

$$\Diamond_X \dashv i_X \dashv \Box_X : \mathrm{Cat}/X \to \mathrm{Set}^{X^{\mathrm{op}}}$$

$$\Diamond'_X \dashv i'_X \dashv \Box'_X : \mathrm{Cat}/X \to \mathrm{Set}^X$$

Reformulation of the complementation laws via interior operator

The complement of a closed part [resp. a discrete opfibration]
relative to an open part [resp. discrete fibration]
is open [resp. a discrete fibration] ,
that is it factors through i_X :

$$\frac{i'_X M \Rightarrow i_X A}{i_X \Box_X (i'_X M \Rightarrow i_X A)}$$

The stability laws

Also important are the following formally similar laws:

Closed parts are closed wrt finite intersections

$$\frac{\top_X}{i'_X \Diamond'_X \top_X} \quad ; \quad \frac{i'_X M \cap i'_X N}{i'_X \Diamond'_X (i'_X M \cap i'_X N)}$$

Discrete opfibrations are closed wrt finite products

$$\frac{\top_X}{i'_X \Diamond'_X \top_X} \quad ; \quad \frac{i'_X M \times_X i'_X N}{i'_X \Diamond'_X (i'_X M \times_X i'_X N)}$$

Dually

Of course, for actions of categories
the dual laws hold :

$$\frac{i_X A \Rightarrow i'_X M}{i'_X \Box'_X (i_X A \Rightarrow i' M)}$$

$$\frac{\top_X}{i_X \Diamond_X \top_X} \quad ; \quad \frac{i_X A \times_X i_X B}{i_X \Diamond_X (i_X A \times_X i_X B)}$$

In fact, they also hold (not only for posets but also)
for topological spaces:
the partially defined coreflection in closed parts $i'_X \dashv \Box'_X$
and reflection in open parts $\Diamond_X \dashv i_X$ are therein defined.

Toward temporal doctrines

We are now in a position to abstract from the indexed inclusions:

$$\mathrm{Set}^{X^{\mathrm{op}}} \to \mathrm{Cat}/X \leftarrow \mathrm{Set}^{X} \qquad ; \qquad X \in \mathrm{Cat}$$

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$$\mathcal{O}X \to \mathcal{P}X \leftarrow \mathcal{C}X \qquad ; \qquad X \in \mathrm{Top}$$

(In fact, topological spaces
fit only a weak form of temporal doctrine)

Temporal doctrines

Definition

A temporal monoidal doctrine

$$\langle i^{\ell}_X : \mathcal{L}X \to \mathcal{P}X \leftarrow \mathcal{R}X : i^{r}_X \ ; \ X \in \mathcal{C} \rangle$$

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and two fully faithful indexed functors

$$i^{\ell}_X : \mathcal{L}X \to \mathcal{P}X \qquad ; \qquad i^{r}_X : \mathcal{R}X \to \mathcal{P}X$$

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$$i^{\ell}_X : \mathcal{L}X \to \mathcal{P}X \qquad ; \qquad i^{r}_X : \mathcal{R}X \to \mathcal{P}X$$

satisfying four axioms which are in fact exactness conditions.

An indexed strong monoidal category...

Before presenting the axioms,
let us be more explicit about the data:

$\mathcal{P}X$ is an indexed strong monoidal category

for any $f : X \to Y$ in $\mathcal{C}$
the substitution functor $f\cdot : \mathcal{P}Y \to \mathcal{P}X$
preserves the monoidal structure:
there are (coherent) isomorphisms

$$\frac{f \cdot \top_Y}{\top_X} \quad ; \quad \frac{f \cdot (P \otimes_Y Q)}{f \cdot P \otimes_X f \cdot Q}$$

that is $\top_X$ and $\otimes_X$ are indexed functors.

...and two fully faithful indexed functors...

Two indexed categories $\mathcal{L}X$ and $\mathcal{R}X$...

for any $f : X \to Y$ in $\mathcal{C}$ we have substitutions:

$$f^\ell : \mathcal{L}Y \to \mathcal{L}X \qquad ; \qquad f^r : \mathcal{R}Y \to \mathcal{R}X$$

...and two indexed inclusions i^ℓ_X and i^r_X

The functors

$$i^\ell_X : \mathcal{L}X \hookrightarrow \mathcal{P}X \qquad ; \qquad i^r_X : \mathcal{R}X \hookrightarrow \mathcal{P}X$$

are fully faithful and commute with substitutions:

$$f \cdot i^\ell_Y \cong i^\ell_X f^\ell \qquad ; \qquad f \cdot i^r_Y \cong i^r_X f^r$$

First axiom for temporal doctrines: $\mathcal{P}X$ has restricted internal homs, preserved by substitutions

Restricted internal homs:
tensoring by a left or a right action
has a right adjoint...

$$- \otimes_X i^{\ell}_X A \dashv i^{\ell}_X A \Rightarrow_X - \qquad ; \qquad - \otimes_X i^{r}_X M \dashv i^{r}_X M \Rightarrow_X -$$

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...which is preserved by substitutions

$$\frac{f \cdot (i^\ell_Y A \Rightarrow_Y P)}{i^\ell_X f^\ell A \Rightarrow_X f \cdot P} \qquad ; \qquad \frac{f \cdot (i^r_Y M \Rightarrow_Y P)}{i^r_X f^r M \Rightarrow_X f \cdot P}$$

Second axiom for temporal doctrines (part one): Temporal operators and the stability laws

For any $X \in \mathcal{C}$, there are adjunctions

Temporal operators: Left and right closure and interior

$$\Diamond^{\ell}_X \dashv i^{\ell}_X \dashv \Box^{\ell}_X \qquad ; \qquad \Diamond^{r}_X \dashv i^{r}_X \dashv \Box^{r}_X$$

Second axiom for temporal doctrines (part one): Temporal operators and the stability laws

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satisfying the laws (isomorphisms):

The stability laws:
Left and right actions are closed wrt tensor product

$$\frac{i^{\ell}_X \Diamond^{\ell}_X \top_X}{\top_X} \qquad ; \qquad \frac{i^{r}_X \Diamond^{r}_X \top_X}{\top_X}$$

$$\frac{i^{\ell}_X \Diamond^{\ell}_X (i^{\ell}_X A \otimes i^{\ell}_X B)}{i^{\ell}_X A \otimes i^{\ell}_X B} \qquad ; \qquad \frac{i^{r}_X \Diamond^{r}_X (i^{r}_X M \otimes i^{r}_X N)}{i^{r}_X M \otimes i^{r}_X N}$$

Second axiom (part two): The complementation laws

For any $X \in \mathcal{C}$ there are isomorphisms:

Complementation laws, first form

$$\frac{i^r_X \Box^r_X (i^\ell_X A \Rightarrow i^r_X M)}{i^\ell_X A \Rightarrow i^r_X M} \quad ; \quad \frac{i^\ell_X \Box^\ell_X (i^r_X M \Rightarrow i^\ell_X A)}{i^r_X M \Rightarrow i^\ell_X A}$$

Equivalently:

Complementation laws, adjoint form

$$\frac{\Diamond^\ell_X (i^\ell_X \Diamond^\ell_X P \otimes i^r_X M)}{\Diamond^\ell_X (P \otimes i^r_X M)} \quad ; \quad \frac{\Diamond^r_X (i^r_X \Diamond^r_X P \otimes i^\ell_X A)}{\Diamond^r_X (P \otimes i^\ell_X A)}$$

Third axiom for temporal doctrines: Quantifications

For any $f : X \to Y$, there are adjunctions

Kan extensions

$$\exists^{\ell}_{f} \dashv f^{\ell} \dashv \forall^{\ell}_{f} \qquad ; \qquad \exists^{r}_{f} \dashv f^{r} \dashv \forall^{r}_{f}$$

Fourth (and last) axiom : 1 is groupoidal

$\mathcal{C}$ has a teminal object 1 and the projections

$$j^\ell : \mathcal{L}1 \times_{\mathcal{P}1} \mathcal{R}1 \to \mathcal{L}1 \qquad ; \qquad j^r : \mathcal{L}1 \times_{\mathcal{P}1} \mathcal{R}1 \to \mathcal{R}1$$

are isomorphisms:

$$\frac{\mathcal{L}1 \times_{\mathcal{P}1} \mathcal{R}1}{\mathcal{L}1} \qquad ; \qquad \frac{\mathcal{L}1 \times_{\mathcal{P}1} \mathcal{R}1}{\mathcal{R}1}$$

Equivalently:

$\mathcal{L}1 \cong \mathcal{R}1$ as subcategories of $\mathcal{P}1$.

Toward mixed doctrines: the direct link between left and right actions

We now present some properties of temporal doctrines which will lead to the further abstraction of

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These are meant to capture
the trace left by a temporal doctrine
on the indexed categories $\mathcal{L}X$ and $\mathcal{R}X$, that is
what remains when we omit the ampler category $\mathcal{P}X$.

The monoidal closed structure on $\mathcal{L}X$ and $\mathcal{R}X$

In a temporal doctrine, by defining:

Left and right tensor product

$$A \otimes^{\ell}_X B := \Diamond^{\ell}_X(i^{\ell}_X A \otimes_X i^{\ell}_X B) \qquad ; \qquad \top^{\ell}_X := \Diamond^{\ell}_X \top_X$$

$$M \otimes^{r}_X N := \Diamond^{r}_X(i^{r}_X M \otimes_X i^{r}_X N) \qquad ; \qquad \top^{r}_X := \Diamond^{r}_X \top_X$$

we get (as a consequence of the stability laws)
monoidal structures on $\mathcal{L}X$ and $\mathcal{R}X$

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$$M \otimes^{r}_{X} N := \Diamond^{r}_{X}(i^{r}_{X} M \otimes_{X} i^{r}_{X} N) \quad ; \quad \top^{r}_{X} := \Diamond^{r}_{X} \top_{X}$$

we get (as a consequence of the stability laws)
monoidal structures on $\mathcal{L}X$ and $\mathcal{R}X$
which are closed:

Internal homs in $\mathcal{L}X$ and $\mathcal{R}X$

$$A \Rightarrow^{\ell}_{X} B := \Box^{\ell}_{X}(i^{\ell}_{X} A \Rightarrow_{X} i^{\ell}_{X} B)$$

$$M \Rightarrow^{r}_{X} N := \Box^{r}_{X}(i^{r}_{X} M \Rightarrow_{X} i^{r}_{X} N)$$

$\mathcal{L}X$ acts on $\mathcal{R}X$
$\mathcal{R}X$ acts on $\mathcal{L}X$

Similarly, by defining:

$$A *^{\ell}_{X} M := \Diamond^{\ell}_{X}(i^{\ell}_{X} A \otimes_X i^{r}_{X} M)$$

$$M *^{r}_{X} A := \Diamond^{r}_{X}(i^{r}_{X} M \otimes_X i^{\ell}_{X} A)$$

we get monoidal actions

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we get monoidal actions
as follows from the stability and complementation laws:

Associativity

$$(A *^\ell_X M) *^\ell_X N := \Diamond^\ell_X(i^\ell_X \Diamond^\ell_X(i^\ell_X A \otimes_X i^r_X M) \otimes_X i^r_X N) \cong$$

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$$\Diamond^{\ell}_X((i^{\ell}_X A \otimes_X i^r_X \Diamond^r_X(i^r_X M \otimes_X i^r_X N)) := A *^{\ell}_X (M \otimes^r_X N)$$

The monoidal actions are biclosed

The functors

$$- *^{\ell}_{X} M : \mathcal{L}X \to \mathcal{L}X \qquad ; \qquad A *^{\ell}_{X} - : \mathcal{R}X \to \mathcal{L}X$$

have right adjoints

$$- *^{\ell}_{X} M \dashv M \succ^{\ell}_{X} - \qquad ; \qquad A *^{\ell}_{X} - \dashv A \triangleright^{r}_{X} -$$

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Relative complement

$$M \succ^{\ell}_{X} A := \Box^{\ell}_{X}(i^{r}_{X} M \Rightarrow_{X} i^{\ell}_{X} A)$$

Skew homs

$$A \rhd^{r}_{X} B := \Box^{r}_{X}(i^{\ell}_{X} A \Rightarrow_{X} i^{\ell}_{X} B)$$

and similarly for the other action $*^{r}_{X}$.

Substitution preserves tensor and complement

Focussing on left actions,

Left substitution preserves left tensor

$$f^\ell \top^\ell_Y \cong \top^\ell_X \qquad ; \qquad f^\ell(A \otimes^\ell_Y B) \cong f^\ell A \otimes^\ell_X f^\ell B$$

Left substitution preserves left complement

$$f^\ell(M \succ^\ell_Y A) \cong f^r M \succ^\ell_X f^\ell A$$

Proof (for complement)

$f^\ell(M \succ^\ell_Y A) := f^\ell \Box^\ell_X (i^r_Y M \Rightarrow_Y i^\ell_Y A) \cong$

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Left substitution preserves left complement

$$f^\ell(M \succ^\ell_Y A) \cong f^r M \succ^\ell_X f^\ell A$$

Proof (for complement)

$f^\ell(M \succ^\ell_Y A) := f^\ell \Box^\ell_X(i^r_Y M \Rightarrow_Y i^\ell_Y A) \cong$
$\Box^\ell_X i^\ell_X f^\ell \Box^\ell_Y(i^r_Y M \Rightarrow_Y i^\ell_Y A) \cong$
$\Box^\ell_X f \cdot i^\ell_Y \Box^\ell_Y(i^r_Y M \Rightarrow_Y i^\ell_Y A) \cong$

Substitution preserves tensor and complement

Focussing on left actions,

Left substitution preserves left tensor

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$$\Box^\ell_X f \cdot (i^r_Y M \Rightarrow_Y i^\ell_Y A) \cong$$
$$\Box^\ell_X (i^r_X f^r M \Rightarrow_X i^\ell_X f^\ell A) := f^r M \succ^\ell_X f^\ell A$$

Truth values and constant actions

By the last axiom of temporal doctrines,

the "truth values" category

$$\mathcal{V}_0 := \mathcal{L}1 \times_{\mathcal{P}1} \mathcal{R}1$$

has equivalences (namely the projections)

$$j^\ell : \mathcal{V}_0 \to \mathcal{L}1 \qquad ; \qquad j^r : \mathcal{V}_0 \to \mathcal{R}1$$

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For a truth value $V \in \mathcal{V}_0$,

the left and right V-constant actions

are obtained by substituting the constant term $X : X \to 1$ in the projections $j^\ell V$ and $j^r V$:

$$X^\ell j^\ell V \qquad ; \qquad X^r j^r V$$

Collapse of operators for V-constant actions

Constant actions laws

There are isomorphisms

$$\frac{A \otimes^{\ell}_{X} X^{\ell} j^{\ell} V}{A *^{\ell}_{X} X^{r} j^{r} V} \quad ; \quad \frac{M \succ^{\ell}_{X} X^{\ell} j^{\ell} V}{M \rhd^{\ell}_{X} X^{r} j^{r} V}$$

Proof

Since $i^{\ell} j^{\ell} = i^{r} j^{r}$ we have

$A \otimes^{\ell}_{X} X^{\ell} j^{\ell} V := \Diamond^{\ell}_{X}(i^{\ell}_{X} A \otimes_{X} i^{\ell}_{X} X^{\ell} j^{\ell} V) \cong$

$\Diamond^{\ell}_{X}(i^{\ell}_{X} A \otimes_{X} X \cdot i^{\ell} j^{\ell} V) \cong$

$\Diamond^{\ell}_{X}(i^{\ell}_{X} A \otimes_{X} X \cdot i^{r} j^{r} V) \cong$

$\Diamond^{\ell}_{X}(i^{\ell}_{X} A \otimes_{X} i^{r}_{X} X^{r} j^{r} V) := A *^{\ell}_{X} X^{r} j^{r} V$

A temporal doctrine originates a mixed doctrine

Summarizing, we have seen that a temporal doctrine
$\langle i^{\ell}_X : \mathcal{L}X \to \mathcal{P}X \leftarrow \mathcal{R}X : i^{r}_X \; ; \; X \in \mathcal{C} \rangle$

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$$\langle i^\ell_X : \mathcal{L}X \to \mathcal{P}X \leftarrow \mathcal{R}X : i^r_X \; ; \; X \in \mathcal{C} \rangle$$

gives rise to a

Mixed doctrine

$$\langle \mathcal{L}X \, , \, \mathcal{R}X \, , \, \top^\ell_X \, , \, \top^r_X \, , \, \otimes^\ell_X \, , \, \otimes^r_X \, , \, *^\ell_X \, , \, *^r_X \; ; \; X \in \mathcal{C} \rangle$$

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$\langle i^\ell_X : \mathcal{L}X \to \mathcal{P}X \leftarrow \mathcal{R}X : i^r_X \; ; \; X \in \mathcal{C} \rangle$

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$\langle \mathcal{L}X \,,\, \mathcal{R}X \,,\, \top^\ell_X \,,\, \top^r_X \,,\, \otimes^\ell_X \,,\, \otimes^r_X \,,\, *^\ell_X \,,\, *^r_X \; ; \; X \in \mathcal{C} \rangle$

$\langle \mathcal{L}X \,,\, \top^\ell_X \,,\, \otimes^\ell_X \rangle \quad ; \quad \langle \mathcal{R}X \,,\, \top^r_X \,,\, \otimes^r_X \rangle$

are symmetric monoidal categories,
indexed by a category $\mathcal{C}$ with a terminal object 1,
which act on each other:

$*^\ell_X : \mathcal{L}X \times \mathcal{R}X \to \mathcal{L}X \quad ; \quad *^r_X : \mathcal{R}X \times \mathcal{L}X \to \mathcal{R}X$

satisfying the following axioms :

First axiom for a mixed doctrine:
The monoidal categories and actions are biclosed

The monoidal categories $\mathcal{L}X$ and $\mathcal{R}X$ are closed,

$$\frac{\mathcal{L}X(A \otimes^{\ell}_{X} B, C)}{\mathcal{L}X(A, B \Rightarrow^{\ell}_{X} C)} \quad ; \quad \frac{\mathcal{R}X(M \otimes^{r}_{X} N, O)}{\mathcal{R}X(M, N \Rightarrow^{r}_{X} O)}$$

First axiom for a mixed doctrine: The monoidal categories and actions are biclosed

The monoidal categories $\mathcal{L}X$ and $\mathcal{R}X$ are closed, and the monoidal actions are biclosed.

$$\frac{\mathcal{L}X(A \otimes^{\ell}_{X} B, C)}{\mathcal{L}X(A, B \Rightarrow^{\ell}_{X} C)} \quad ; \quad \frac{\mathcal{R}X(M \otimes^{r}_{X} N, O)}{\mathcal{R}X(M, N \Rightarrow^{r}_{X} O)}$$

$$\frac{\mathcal{L}X(A *^{\ell}_{X} M, B)}{\mathcal{L}X(A, M \succ^{\ell}_{X} B)} \quad ; \quad \frac{\mathcal{R}X(M *^{r}_{X} A, N)}{\mathcal{R}X(M, A \succ^{r}_{X} N)}$$

$$\frac{\mathcal{L}X(A *^{\ell}_{X} M, B)}{\mathcal{R}X(M, A \rhd^{r}_{X} B)} \quad ; \quad \frac{\mathcal{R}X(M *^{r}_{X} A, N)}{\mathcal{L}X(A, M \rhd^{\ell}_{X} N)}$$

Second axiom for a mixed doctrine: Substitution functors

f^ℓ and f^r have quantifications,

$$\exists_f^\ell \dashv f^\ell \dashv \forall_f^\ell \qquad ; \qquad \exists_f^r \dashv f^r \dashv \forall_f^r$$

Second axiom for a mixed doctrine: Substitution functors

f^ℓ and f^r have quantifications,
are strong monoidal

$$\exists_f^\ell \dashv f^\ell \dashv \forall_f^\ell \qquad ; \qquad \exists_f^r \dashv f^r \dashv \forall_f^r$$

$$\frac{f^\ell \top_Y^\ell}{\top_X^\ell} \; ; \; \frac{f^\ell(A \otimes_Y^\ell B)}{f^\ell A \otimes_X^\ell f^\ell B} \qquad ; \qquad \frac{f^r \top_Y^r}{\top_X^r} \; ; \; \frac{f^r(M \otimes_Y^r N)}{f^r M \otimes_X^r f^\ell N}$$

Second axiom for a mixed doctrine: Substitution functors

f^ℓ and f^r have quantifications,
are strong monoidal and preserve complements.

$$\exists^\ell_f \dashv f^\ell \dashv \forall^\ell_f \qquad ; \qquad \exists^r_f \dashv f^r \dashv \forall^r_f$$

$$\frac{f^\ell \top^\ell_Y}{\top^\ell_X} \; ; \; \frac{f^\ell (A \otimes^\ell_Y B)}{f^\ell A \otimes^\ell_X f^\ell B} \qquad ; \qquad \frac{f^r \top^r_Y}{\top^r_X} \; ; \; \frac{f^r (M \otimes^r_Y N)}{f^r M \otimes^r_X f^\ell N}$$

$$\frac{f^\ell (M \succ^\ell_Y A)}{f^r M \succ^\ell_X f^\ell A} \qquad ; \qquad \frac{f^r (A \succ^r_Y M)}{f^\ell A \succ^r_X f^r M}$$

Third and last axiom for a mixed doctrine: Truth values and constant actions

There is a "truth values" category $\mathcal{V}_0$ with equivalences

$$j^{\ell} : \mathcal{V}_0 \to \mathcal{L}1 \qquad ; \qquad j^{r} : \mathcal{V}_0 \to \mathcal{R}1$$

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There is a "truth values" category $\mathcal{V}_0$ with equivalences

$$j^\ell : \mathcal{V}_0 \to \mathcal{L}1 \qquad ; \qquad j^r : \mathcal{V}_0 \to \mathcal{R}1$$

satisfying the constant actions laws:

$$\frac{A \otimes^\ell_X X^\ell j^\ell V}{A *^\ell_X X^r j^r V} \qquad ; \qquad \frac{M \succ^\ell_X X^\ell j^\ell V}{M \triangleright^\ell_X X^r j^r V}$$

$$\frac{M \otimes^r_X X^r j^r V}{M *^r_X X^\ell j^\ell V} \qquad ; \qquad \frac{A \succ^r_X X^r j^r V}{A \triangleright^r_X X^\ell j^\ell V}$$

The copower and negation functor

The above laws give the “copower” and “negation” functors:

Left copower

$$A \bullet^{\ell}_X V := \frac{A \otimes^{\ell}_X X^{\ell} j^{\ell} V}{A *^{\ell}_X X^{r} j^{r} V}$$

Right copower

$$M \bullet^{r}_X V := \frac{M \otimes^{r}_X X^{r} j^{r} V}{M *^{r}_X X^{\ell} j^{\ell} V}$$

Left negation

$$\neg^{\ell}_X (M, V) := \frac{M \succ^{\ell}_X X^{\ell} j^{\ell} V}{M \rhd^{\ell}_X X^{r} j^{r} V}$$

Right negation

$$\neg^{r}_X (A, V) := \frac{A \succ^{r}_X X^{r} j^{r} V}{A \rhd^{r}_X X^{\ell} j^{\ell} V}$$

The mixed tensor product

We have adjunctions with parameter:

$$\underline{\mathrm{ten}}_X(A, M) \dashv_M \neg^\ell_X(M, V) \qquad ; \qquad \underline{\mathrm{ten}}_X(A, M) \dashv_A \neg^r_X(A, V)$$

Mixed tensor product

$$\underline{\mathrm{ten}}_X(A, M) := \frac{j^\ell_- \exists^\ell_X (A *^\ell_X M)}{j^r_- \exists^r_X (M *^r_X A)}$$

(where $j^\ell_- : \mathcal{L}1 \to \mathcal{V}_0$ is an equivalence adjoint to j^ℓ).

Powers

We have adjunctions with parameter:

$$A \bullet^{\ell}_{X} V \dashv_{V} [V, B]^{\ell}_{X} \qquad ; \qquad M \bullet^{r}_{X} V \dashv_{V} [V, M]^{r}_{X}$$

Left power

$$[V, A]^{\ell}_{X} := \frac{X^{\ell} j^{\ell} V \Rightarrow^{\ell}_{X} A}{X^{r} j^{r} V \succ^{\ell}_{X} A}$$

Right power

$$[V, M]^{r}_{X} := \frac{X^{r} j^{r} V \Rightarrow^{r}_{X} M}{X^{\ell} j^{\ell} V \succ^{r}_{X} M}$$

Natural transformations

We have adjunctions with (another) parameter:

$$A \bullet^{\ell}_{X} V \dashv_{A} \underline{\mathrm{nat}}^{\ell}_{X}(A, B) \qquad ; \qquad M \bullet^{r}_{X} V \dashv_{M} \underline{\mathrm{nat}}^{r}_{X}(M, N)$$

Left natural transformations

$$\underline{\mathrm{nat}}^{\ell}_{X}(A, B) := \frac{j^{\ell}_{-} \forall^{\ell}_{X} (A \Rightarrow^{\ell}_{X} B)}{j^{r}_{-} \forall^{r}_{X} (A \rhd^{r}_{X} B)}$$

Right natural transformations

$$\underline{\mathrm{nat}}^{r}_{X}(A, B) := \frac{j^{r}_{-} \forall^{r}_{X} (M \Rightarrow^{r}_{X} N)}{j^{\ell}_{-} \forall^{\ell}_{X} (M \rhd^{\ell}_{X} N)}$$

Left and right actions are enriched in truth values by natural transformations

The monoidal structures on $\mathcal{L}1$ and $\mathcal{M}1$ induce, via j^{ℓ}_{-} and j^{r}_{-} the same monoidal structure $\mathcal{V}$ on $\mathcal{V}_0$.

Left and right actions are enriched in truth values by natural transformations

The monoidal structures on $\mathcal{L}1$ and $\mathcal{M}1$ induce, via j^ℓ_- and j^r_- the same monoidal structure $\mathcal{V}$ on $\mathcal{V}_0$.

All the $\mathcal{L}X$ and $\mathcal{R}X$ are enriched over $\mathcal{V}$, by

$$\underline{\mathrm{nat}}^\ell_X(A, B) \qquad ; \qquad \underline{\mathrm{nat}}^r_X(M, N)$$

They are also tensored and cotensored over $\mathcal{V}$, via the power and copower functors.

Each substitution functor $f^\ell : \mathcal{L}Y \to \mathcal{L}X$ and $f^r : \mathcal{R}Y \to \mathcal{R}X$ is in fact also enriched over $\mathcal{V}$.

Monoidal actions and enrichments

In fact, from general facts on monoidal actions
(see the TAC 2001 paper by Janelidze & Kelly)
and the first axiom of mixed doctrines, it follows that

the categories $\mathcal{L}X$ of left actions are enriched,
tensored (have copowers) and cotensored (have powers)
over themselves and over $\mathcal{R}X$ (and conversely).

The constant actions laws then assure that
by applying the quantifications $\forall^{\ell}_{X}$ and $\forall^{r}_{X}$,
we obtain enrichings over $\mathcal{L}1$ and $\mathcal{R}1$
which coincide via the equivalence $j^r j^{\ell}_{-} : \mathcal{L}1 \to \mathcal{R}1$.

Substitutions preserve powers, copowers and negation

Since substitutions preserve tensor $\otimes_X$ and complement $\succ_X$, they also preserve powers, copowers and nagation:

$$\frac{f^\ell(A \bullet^\ell_Y V)}{f^\ell A \bullet^\ell_X V} \quad ; \quad \frac{f^\ell [V, A]^\ell_Y}{[V, f^\ell A]^\ell_X} \quad ; \quad \frac{f^\ell \neg^\ell_Y (M, V)}{\neg^\ell_X (f^r M, V)}$$

(and similarly for right substitution).
From the above laws we obtain, by adjunction, some more laws:

Nine laws (the first three)

The fact that f^ℓ preserves copowers is equivalent
the fact that the adjunction $f^\ell \dashv \forall^\ell_f$ is enriched in $\mathcal{V}$

$$\frac{f^\ell(A \bullet^\ell_Y V)}{f^\ell A \bullet^\ell_X V} \dashv_A \frac{\underline{\text{nat}}_Y(A, \forall^\ell_f B)}{\underline{\text{nat}}_X(f^\ell A, B)}$$

Nine laws (the first three)

The fact that f^ℓ preserves copowers is equivalent the fact that the adjunction $f^\ell \dashv \forall_f^\ell$ is enriched in $\mathcal{V}$

$$\frac{f^\ell(A \bullet_Y^\ell V)}{f^\ell A \bullet_X^\ell V} \dashv_A \frac{\underline{\mathrm{nat}}_Y(A, \forall_f^\ell B)}{\underline{\mathrm{nat}}_X(f^\ell A, B)}$$

and that $\forall_f^\ell$ preserves powers

$$\frac{f^\ell(A \bullet_Y^\ell V)}{f^\ell A \bullet_X^\ell V} \dashv_V \frac{[V, \forall_f^\ell B]_Y^\ell}{\forall_f^\ell [V, B]_X^\ell}$$

Nine laws (the middle three)

The fact that f^ℓ preserves powers is equivalent to the fact that to the fact that $\exists_f^\ell$ preserves copowers

$$\frac{(\exists_f^\ell A) \bullet_Y^\ell V}{\exists_f^\ell (A \bullet_X^\ell V)} \dashv_V \frac{f^\ell [V, B]_Y^\ell}{[V, f^\ell B]_X^\ell}$$

Nine laws (the middle three)

The fact that f^ℓ preserves powers is equivalent to the fact that to the fact that $\exists^\ell_f$ preserves copowers

$$\frac{(\exists^\ell_f A) \bullet^\ell_Y V}{\exists^\ell_f (A \bullet^\ell_X V)} \dashv_V \frac{f^\ell [V, B]^\ell_Y}{[V, f^\ell B]^\ell_X}$$

and that the adjunction $\exists^\ell_f \dashv f^\ell$ is enriched in $\mathcal{V}$

$$\frac{(\exists^\ell_f A) \bullet^\ell_Y V}{\exists^\ell_f (A \bullet^\ell_X V)} \dashv_A \frac{\underline{\mathrm{nat}}_Y (\exists^\ell_f A, B)}{\underline{\mathrm{nat}}_X (A, f^\ell B)}$$

Nine laws (the last three)

The fact that f^ℓ preserves negations is equivalent to a law relating $\exists_f^\ell$ and the mixed tensor product

Negation and quantifications

$$\frac{\underline{\mathrm{ten}}_Y(\exists_f^\ell A, M)}{\underline{\mathrm{ten}}_X(A, f^\ell M)} \dashv_M \frac{f^\ell \neg_Y^\ell(M, V)}{\neg_X^\ell(f^r M, V)}$$

Nine laws (the last three)

The fact that f^ℓ preserves negations is equivalent to a law relating $\exists^\ell_f$ and the mixed tensor product

Negation and quantifications

$$\frac{\underline{\text{ten}}_Y(\exists^\ell_f A, M)}{\underline{\text{ten}}_X(A, f^\ell M)} \dashv_M \frac{f^\ell \neg^\ell_Y(M, V)}{\neg^\ell_X(f^r M, V)}$$

and to a well-known law relating negation and quantifications

$$\frac{\underline{\text{ten}}_Y(\exists^\ell_f A, M)}{\underline{\text{ten}}_X(A, f^\ell M)} \dashv_A \frac{\neg^r_Y(\exists^\ell_f A, V)}{\forall^r_f \neg^r_X(A, V)}$$

All the adjunctions are enriched in $\mathcal{V}$

Along with the quantification adjunctions,
also the adjunctions defining

$$\Rightarrow^{\ell}_{X}\ ;\ \Rightarrow^{r}_{X} \qquad ; \qquad \succ^{\ell}_{X}\ ;\ \succ^{r}_{X} \qquad ; \qquad \triangleright^{\ell}_{X}\ ;\ \triangleright^{r}_{X}$$

turn out to be enriched in $\mathcal{V}$:

All the adjunction defining a mixed doctrine
and thus also the derived ones such as

$$\underline{\text{ten}}_X(A, M) \dashv_A \neg^{\ell}_X(M, V)$$

are enriched in $\mathcal{V}$.

What we can do with mixed doctrines? Just one instance: the Yoneda Lemmas

Given a "point" $x : 1 \to X$, its "left-image" $\exists^{\ell}_{x} \top^{\ell}_{1}$ gives, for $\mathcal{C} = \mathrm{Cat}$, the presheaf represented by the object x.

Yoneda and co-Yoneda Lemmas are instances of two of the nine laws

$$\dfrac{\dfrac{\underline{\mathrm{nat}}^{\ell}_{X}(\exists^{\ell}_{x} \top^{\ell}_{1}, A)}{\underline{\mathrm{nat}}^{\ell}_{1}(\top^{\ell}_{1}, x^{\ell} A)}}{j^{\ell}_{-} x^{\ell} A} \quad ; \quad \dfrac{\dfrac{\underline{\mathrm{ten}}_{X}(\exists^{\ell}_{x} \top^{\ell}_{1}, M)}{\underline{\mathrm{ten}}_{1}(\top^{\ell}_{1}, x^{r} M)}}{j^{r}_{-} x^{r} M}$$

Concluding remarks

“Topology applies to categories”

The abstract frame of temporal and mixed doctrines, sometimes with a further “comprehension” axiom relating $\mathcal{L}X$ and $\mathcal{R}X$ with $\mathcal{C}/X$, turns out to be a rather effective tool for treating some aspects of category theory [Pisani, 2010].

One open question: range of the theory

For which monoidal closed categories $\mathcal{V}$ do the categories of left and right $\mathcal{V}$-actions constitute a mixed doctrine (as for $\mathcal{V} = \mathrm{Set}$ and $\mathcal{V} = 2$)?