Algebraic Theories over Nominal Sets

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June 21, 2010

- joint work with Alexander Kurz and Jiří Velebil -

Abstract syntax with variable binding

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The initial algebra view of syntax still works if one moves from Set to nominal sets (Gabay-Pitts, 1999)

The category Nom of nominal sets

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A nominal set $(X, \cdot)$ is a left action of $\mathfrak{S}(\mathcal{N})$, such that all elements of X have a finite support. Morphisms are ‘equivariant’ maps.

Nominal sets and $[\mathbb{I}, \mathrm{Set}]$

Consider the category $\mathbb{I}$ with objects finite subsets of $\mathcal{N}$ and morphisms injective maps

$|\mathbb{I}|$ denotes the underlying discrete category of $\mathbb{I}$.

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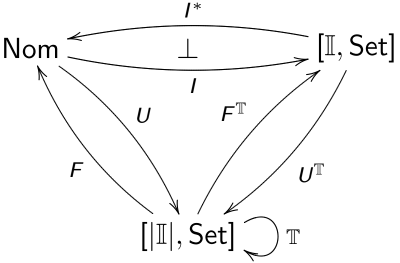

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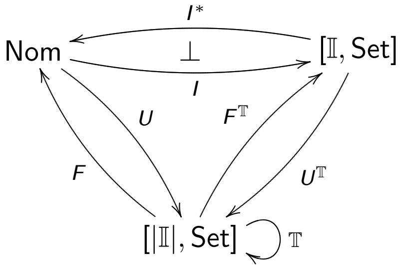
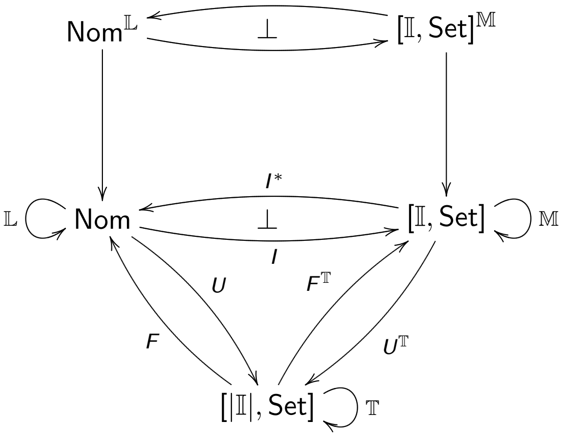


$(U\mathbb{X})(S) = \{x \in X \ : \ S \text{ supports } x\}$
The adjunction $F \dashv U$ is of descent type.
The comparison functor I is full, faithful, preserves filtered colimits, coproducts, but does not preserve epis.

Monads over nominal sets

What are the monads that can be transported via the adjunction $I^* \dashv I$?



Monads over Set^{Srt}

- For any monad M on Set^{Srt} there is a (possibly proper) class Σ of (possibly infinitary) operations and a class E of equations such that

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- For any finitary monad M on Set^{Srt} there is a set Σ of finitary operations and a set E of equations in finitely many variables such that

$$\mathsf{Alg}(M) \cong \mathsf{Alg}(\Sigma, E)$$

  Conversely, for any such (Σ, E), there is a finitary monad M such that the above holds.

Monads over l.f.p categories

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Kelly and Power proved that for every finitary monad $\mathbb{T}$ on $\mathscr{K}$, there exist two finitary signatures Γ and Σ and a coequalizer diagram

$$\mathbb{F}_\Gamma \rightrightarrows \mathbb{F}_\Sigma \longrightarrow \mathbb{T}$$

in the category of finitary monads on $\mathscr{K}$, where $\mathbb{F}_\Gamma$ and $\mathbb{F}_\Sigma$ are free (finitary) monads on Γ and Σ, respectively.

Finitary based signatures, functors and monads

Consider an adjunction $F \dashv U : \mathscr{K} \to \mathscr{X}$ of descent type, ie we have coequalizers

$$FUFUA \underset{FU\varepsilon A}{\overset{\varepsilon FUA}{\rightrightarrows}} FUA \xrightarrow{\varepsilon A} A \tag{1}$$

Definition

A fb-signature on $\mathscr{K}$ is a family Σn of objects of $\mathscr{K}$, indexed by f.p. objects n in $\mathscr{X}$.

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Definition

A functor $L : \mathscr{K} \to \mathscr{K}$ is called based if L preserves all coequalizers of type (1).

A monad $\mathbb{M} = (M, \mu, \eta)$ on $\mathscr{K}$ is called based if M is a based functor.

A finitary and based functor/monad is called an fb-functor/monad.

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Finitary based functors and monads

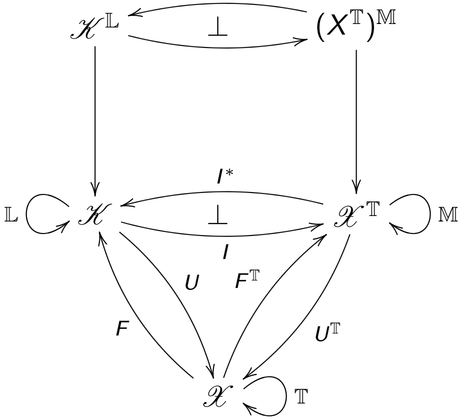
Theorem (Kurz, Velebil)

Fb-functors/monads are precisely those functors/monads on $\mathscr{K}$ that can be presented by operations taking arities from f.p. objects of $\mathscr{X}$.

Theorem (Kurz, Velebil)

If $\mathbb{T}$ is a monad on $\mathscr{X}$ and $\mathbb{M}$ is a fb monad on $\mathscr{X}^{\mathbb{T}}$ then $U : (\mathscr{X}^{\mathbb{T}})^{\mathbb{M}} \to \mathscr{X}$ is monadic.

Transporting finitary based functors and monads



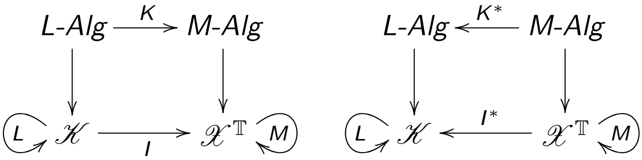
Theorem

If $\mathbb{L}$ is a fb monad, then $\mathbb{M} = I\mathbb{L}I^*$ is a fb monad. Moreover, the functor $I{-}I^* : \mathrm{fbMnd}(\mathcal{K}) \to \mathrm{fbMnd}(\mathcal{X}^{\mathbb{T}})$ *has a left-adjoint.*

Transporting finitary based functors and monads

Theorem

Consider a fb-functor/monad L on $\mathscr{K}$ and let $M = ILI^*$ be its "transport along I". Then there are diagrams



commuting up to isomorphism, the left-hand one being a pseudopullback. Moreover, $K^* \dashv K$ holds.

Back to nominal sets

Theorem

Any fb-monad/functor L on Nom *induces a UA-theory Φ on* $[\mathbb{I}, \mathrm{Set}]$*, so that the category of L-algebras is the category of Φ-algebras 'restricted along I'.*

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This allows us to transfer universal algebra results from $[\mathbb{I}, \mathsf{Set}]$ to Nom.

Definition

Let L be a functor on Nom. A full subcategory $\mathcal{C}$ of L-algebras is equationally definable by a UA-theory Φ on $\mathbb{I}$ if $\mathcal{C}$ consists of L-algebras (A, a) with $K(A, a) \models \Phi$, where $K : L\text{-}Alg \to \Phi\text{-}Alg$ is the lifting of I.

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Theorem

Let L be a fb-functor/monad on Nom*. A class of L-algebras is equationally definable if and only if it is closed under homomorphic images of support-preserving maps, under subalgebras and under products.*

Theorem

Let L be a fb-functor/monad on Nom*. A class of L-algebras is definable by implications if and only if it is closed under subalgebras, products and filtered colimits.*

Other nominal logics

A natural question to ask is how does the equational logic obtained form fb-monads on Nom compare to other 'nominal' logics.

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Nominal algebra, developed by Gabbay and Mathijssen, and NEL, developed by Clouston and Pitts, fit in the general framework developed here. For example, the signatures of nominal algebra are given by functors of the form $\mathcal{N} + [\mathcal{N}] + \Sigma$, where $\mathcal{N}$ is the constant functor, $[\mathcal{N}]$ is the abstraction functor and Σ is a polynomial functor. These functors are uniform and finitary based.

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A UA-theory of $[\mathbb{I}, \mathrm{Set}]$ corresponding to $\mathbb{T}$

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operation symbols:

$$\begin{array}{ll} (b/a)_S : S \cup \{a\} \to S \cup \{b\} & a \neq b,\ a \notin S,\ b \notin S \\ w_{S,a} : S \to S \cup \{a\} & a \notin S \end{array}$$

equations $E_{[\mathbb{I},\mathrm{Set}]}$:

$$\begin{array}{l} (a/b)_S(b/a)_S(x) = x \\ (b/a)_{S\cup\{d\}}(d/c)_{S\cup\{a\}}(x) = (d/c)_{S\cup\{b\}}(b/a)_{S\cup\{c\}}(x) \\ (c/b)_S(b/a)_S(x) = (c/a)_S \\ (b/a)_{S\cup\{c\}}w_{S\cup\{a\},c}(x) = w_{S\cup\{b\},c}(b/a)_S \\ (b/a)_S w_{S,a}(x) = w_{S,b}(x) \\ w_{S\cup\{b\},a}w_{S,b}(x) = w_{S\cup\{a\},b}w_{S,a}(x) \end{array}$$

Uniform theories via an example

UA-presentation of the abstraction functor δ.

operation symbols Op_δ:
$[a]_S : S \cup \{a\} \to S$
for all finite sets S and $a \notin S$

equations E_δ:

$$
\begin{aligned}
&(c/b)_S [a]_{S\cup\{b\}} t = [a]_{S\cup\{c\}} (c/b)_{S\cup\{a\}} t \\
&[a]_S t = [b]_S (b/a)_S t \\
&w_{S,b} [a]_S t = [a]_{S\cup\{b\}} w_{S\cup\{a\},b} t
\end{aligned}
$$

Uniform theories via an example

Uniform presentation of δ

The operations can be structured as a presheaf:

$$
\begin{array}{l}
[a]_S \in Op(S \cup \{a\}) \\
w_b \cdot [a]_S = [a]_{S \cup \{b\}} \\
(b/a)_S \cdot [a]_S = [b]_S
\end{array}
$$

The equations are uniformly generated by

$$
\begin{array}{l}
(c/b)[a]_{\{b\}} t = [a]_{\{c\}} (c/b)_{\{a\}} t \\
[a]_\emptyset t = [b]_\emptyset (b/a) t \\
w_b [a]_\emptyset t = [a]_{\{b\}} w_{\{a\},b} t
\end{array}
$$

Uniform algebraic theories

Restricting to uniform equational theories corresponds semantically to closure under abstraction.

Theorem

A class of nominal algebras is definable by uniform equations iff it is closed under homomorphic images, subalgebras, products, and abstraction.

This gives a new proof for Gabbay's HSPA-theorem.

Theorem

A class of algebras over nominal sets is definable by uniform implications if and only if it is closed under subalgebras, products, filtered colimits and abstraction.

Thank you!