on the duality between disks and trees

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background

André Joyal defined the category *Disk* and implicitly its opposite category Θ in order to define weak higher categories.

An explicit definition of Θ in terms of trees and (strict) ω-categories was provided by Michael Batanin and Ross Street.

Michael Makkai and Marek Zawadowski and then Clemens Berger provided demonstrations of the duality between *Disk* and Θ.

overview

Here we demonstrate the duality between *Disk* and Θ
by lifting the well known duality between ordinals and intervals.

We do so by extending the category of ordinals and the category of intervals using induction and also using so-called labeled trees.

The extended categories of intervals are equivalent to *Disk*.

The extended categories of ordinals are equivalent to Θ.

ordinals and intervals

Δ_+ — the category of ordinals

- objects: finite ordinals $[n] = \{0, \ldots, n\}$ for $n = -1, 0, \ldots$
- morphisms: order preserving functions

$\mathcal{I}_+$ — the category of intervals

- objects: non-empty ordinals
- morphisms: greatest, least and order preserving functions

There is a well known equivalence $\mathcal{I}_+^{op} \cong \Delta_+$ by the functors

$$(\_)^\wedge : \Delta_+^{op} \to \mathcal{I}_+ \qquad (\_)^\vee : \mathcal{I}_+^{op} \to \Delta_+$$
$$: [n] \mapsto [n+1] \qquad : [n] \mapsto [n-1].$$

Next we introduce categories named $i\mathcal{I}_+$ and $i\Delta_+$ which are inductively defined extensions of $\mathcal{I}_+$ and Δ_+.

the category $i\mathcal{I}_+$

$i\mathcal{I}_+$ — an extension of the intervals defined inductively

The objects of $i\mathcal{I}_+$ are graded by height.
The interval $[0]$ is the trivial object of $i\mathcal{I}_+$ and has height 0 (zero).

H, an object of height n, consists of

- an interval $\operatorname{Ob} H$, called the root object of H
- an object $H(i)$ of $i\mathcal{I}_+$ of height $< n$ for each $i \in \operatorname{Ob} H$ which is trivial if and only if i is an endpoint

A morphism $g \colon H \to H'$ consists of

- an interval morphism $g \colon \operatorname{Ob} H \to \operatorname{Ob} H'$
- a morphism $g(i) \colon H(i) \to H'(gi)$ for each $i \in \operatorname{Ob} H$

the category $i\Delta_+$

$i\Delta_+$ — an extension of the ordinals defined inductively

The objects of $i\Delta_+$ are graded by height.
The ordinal $[-1]$ is the trivial object of $i\Delta_+$ and has height 0 (zero).

K, an object of height n, consists of

- an ordinal $\mathrm{Ob}\, K$, called the root object of K
- an object $K(j)$ of $i\Delta_+$ of height $< n$ for each $j \in (\mathrm{Ob}\, K)^\wedge$ which is trivial if and only if j is an endpoint

A morphism $g\colon K' \to K$ consists of

- an ordinal morphism $g\colon \mathrm{Ob}\, K' \to \mathrm{Ob}\, K$
- a morphism $g(j)\colon K'(g^\wedge j) \to K(j)$ for each $j \in (\mathrm{Ob}\, K)^\wedge$

equivalence between $i\mathcal{I}_+{}^{op}$ and $i\Delta_+$

The isomorphism

$$\mathcal{I}_+^{op} \cong \Delta_+$$

lifts to an isomorphism

$$i\mathcal{I}_+{}^{op} \cong i\Delta_+$$

using induction on the height of objects.

Next we

- define the category of trees
- recall the definition of Joyal's category *Disk*
- sketch the constructions used to demonstrate $Disk \simeq i\mathcal{I}$

the category of trees

A tree (A, p) is a functor $A\colon \omega^{op} \to Set$

$$\cdots \xrightarrow{p_2} A_2 \xrightarrow{p_1} A_1 \xrightarrow{p_0} A_0$$

where A_0 is a singleton set containing the root of A.
A tree (A, p) has degree n when $A_m \cong A_n$ for all $m \geq n$.

The tree over i for $i \in A_n$ is the largest subtree with root i.

A morphism $g\colon A \to B$ of trees is a natural transformation

$$\begin{array}{ccccccc}
\cdots & \xrightarrow{p_2} & A_2 & \xrightarrow{p_1} & A_1 & \xrightarrow{p_0} & A_0 \\
\cdots & & \downarrow{g_2} & & \downarrow{g_1} & & \downarrow{g_0} \\
\cdots & \xrightarrow[q_2]{} & B_2 & \xrightarrow[q_1]{} & B_1 & \xrightarrow[q_0]{} & B_0.
\end{array}$$

the categories *Disk* and $Disk_+$

A <u>disk</u> as defined by Joyal is a tree (A, p) of finite degree

1. such that the fibers of $p_n \colon A_{n+1} \to A_n$ have interval structure
2. with sections $d_0, d_1 \colon A_n \to A_{n+1}$ of p_n
   where $p_n^*(x) = [d_0(x), .., d_1(x)]$
3. such that the equalizer of $d_0, d_1 \colon A_n \to A_{n+1}$
   is $d_0(A_{n-1}) \cup d_1(A_{n-1})$

A disk morphism is a tree morphism with components that are interval morphisms.

$Disk_+$ includes, in addition, the trees of degree 0 (zero). Such a tree is terminal in $Disk_+$.

equivalence between $Disk_+$ and $i\mathcal{I}_+$

Given a disk (A, p) we construct inductively an object H of $i\mathcal{I}_+$

- $\mathrm{Ob}\, H$ is the interval fiber $p_0^*(x)$ over the root of A
- for every $i \in \mathrm{Ob}\, H$ let $H(i)$ be the object of $i\mathcal{I}_+$ constructed from the disk over i

Given a disk morphism $f\colon (A, p) \to (B, q)$
we construct inductively a morphism $g\colon H \to K$ of $i\mathcal{I}_+$

- $g\colon \mathrm{Ob}\, H \to \mathrm{Ob}\, K$ is the interval morphism $p_0^*(x) \to q_0^*(y)$ between the fibers over the roots of A and B
- for every $i \in \mathrm{Ob}\, H$ then $g(i)$ is the morphism of $i\mathcal{I}_+$ constructed from the restriction of f to the disk over i

We have equivalences $Disk_+ \simeq i\mathcal{I}_+$ and $Disk \simeq i\mathcal{I}$
that are surjective on objects.

objects of Θ

Batanin and Street defined Θ to be the full subcategory of ωCat consisting of the free ω-categories on globular cardinals.

We will soon define so-called ordinal graphs, which are inductively defined counterparts to globular cardinals, and use a similar method to construct objects of Θ.

To construct an object of Θ from an object of $i\Delta_+$ we define

- the category of ordinal graphs
- the free ω-category of an ordinal graph
- a map from objects of $i\Delta_+$ to ordinal graphs

but first we recall the definition of ω-category.

ω-categories

an enriched category $\mathcal{C}$ over a monoidal category $\mathcal{V}$ is

- a set of objects $\mathrm{Ob}\,\mathcal{C}$
- a hom-object $\mathcal{C}(A, B)$ in $\mathcal{V}$ for every pair $A, B \in \mathrm{Ob}\,\mathcal{C}$
- with additional morphisms satisfying certain requirements

strict *n*-categories

- a 1-category is an ordinary category and is enriched over *Set*
- an *n*-category is a category enriched over $(n-1)$-categories

ωCat, the category of *ω-categories*, is the colimit of the diagram

$$0\text{-}Cat \to 1\text{-}Cat \to 2\text{-}Cat \to \cdots \to n\text{-}Cat \to \cdots$$

of inclusions.

the category of ordinal graphs

OGraph — the category of ω-graphs defined inductively

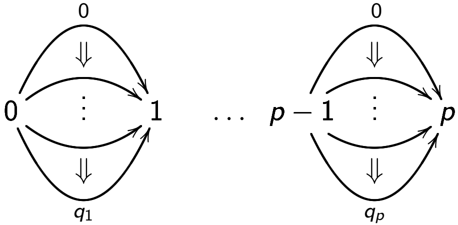
The objects of *OGraph* are graded by dimension.
The ordinal graph of dimension -1 (minus one) is the empty set.

An ordinal graph $\mathcal{G}$ of dimension n consists of

- a finite linearly ordered set $\mathrm{Ob}\,\mathcal{G} = \{x_0, \ldots, x_p\}$ of vertices
- a non-empty ordinal graph $\mathcal{G}(x_{i-1}, x_i)$ (an edge-object of $\mathcal{G}$) of dimension $< n$ for all $i \in \{1, \ldots, p\}$
- $\mathcal{G}(x_i, x_j)$ is empty when $j \neq i + 1$

ordinal graphs and globular cardinals

The ordinal graph $[p]([q_1], \ldots, [q_p])$ is drawn



This ordinal graph corresponds to a globular cardinal.
In fact, we have an equivalence between

- the category of ordinal graphs
- the category of globular cardinals

free ω-categories

The free ω-category $\mathfrak{F}\mathcal{G}$ on an ordinal graph $\mathcal{G}$ has

- objects: those of $\mathcal{G}$, i.e. $x_0, \ldots, x_p$
- hom-objects: defined by induction

$$(\mathfrak{F}\mathcal{G})(x_i, x_j) = \mathfrak{F}(\mathcal{G}(x_{j-1}, x_j)) \times \ldots \times \mathfrak{F}(\mathcal{G}(x_i, x_{i+1}))$$

The free ω-category on an ordinal graph and
the free ω-category on the corresponding globular cardinal
are isomorphic.

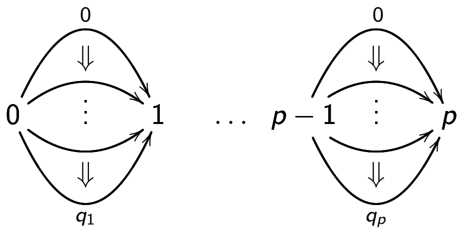
Objects of Δ are free categories on ordinal graphs of dimension 1.

from objects of $i\Delta_+$ to ordinal graphs

An object of $i\Delta_+$ of height 2 (two) can be written

$$[p]([-1], [q_1], \ldots, [q_p], [-1])$$

and corresponds to the ordinal graph



which is written

$$[p]([q_1], \ldots, [q_p])$$

and we have a map $\Upsilon \colon \operatorname{Ob} i\Delta_+ \to OGraph$.

from $i\Delta_+$ to ωCat (on objects)

We have a map

$$\mathrm{Ob}\, i\Delta_+ \xrightarrow{\Upsilon} OGraph \xrightarrow{\mathfrak{F}} \omega Cat$$

which sends H, an object of $i\Delta_+$, to the ω-category $\mathcal{C} = \mathcal{F}(H)$

- objects: elements of $\mathrm{Ob}\, H$ (an ordinal)
- hom-categories: $\mathcal{C}(i,j)$ is the ω-category

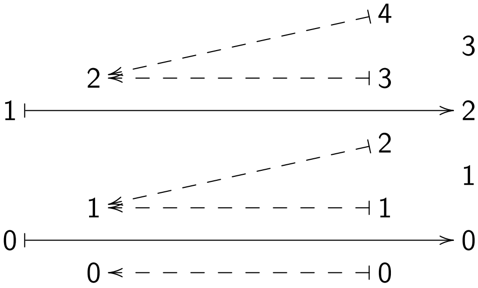
$$\prod_{\ell=i+1}^{j} \mathcal{F}\Upsilon H(\ell)$$

Next we construct an ω-functor $G\colon \mathcal{F}(K') \to \mathcal{F}(K)$ given a morphism $g\colon K' \to K$ of $i\Delta_+$.

the ordinal/interval duality on morphisms

$g\colon K' \to K$ is a morphism of $i\Delta_+$

- its object (ordinal) map $g\colon [1] \to [3]$ is drawn with $\to$
- the dual interval map $g^\wedge\colon [4] \to [2]$ is drawn with $\leftarrow$- -



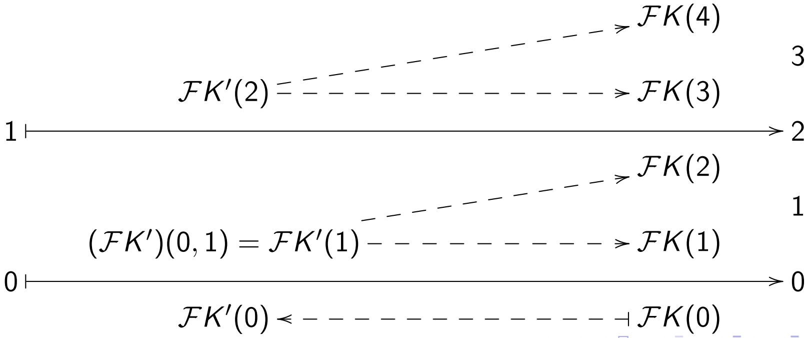
In general, we have $g(j)\colon K'(g^\wedge j) \to K(j)$ for each $j \in (\mathrm{Ob}\,K)^\wedge$.
In particular, we have the two morphisms of $i\Delta_+$

$$g(1)\colon K'(1) \to K(1) \quad \text{and} \quad g(2)\colon K'(1) \to K(2).$$

from $i\Delta_+$ to Θ (on morphisms)

We construct an ω-functor $G\colon \mathcal{F}(K') \to \mathcal{F}(K)$ from $g\colon K' \to K$

- the object map of G is drawn below with $\to$
- by induction we have ω-functors drawn with $\dashrightarrow$
- the ω-functors out of $(\mathcal{F}K')(0,1)$ induce an ω-functor into the product ω-category $\mathcal{F}K(2) \times \mathcal{F}K(3)$ which is the hom-category $(\mathcal{F}K)(G0, G1)$



equivalence between $i\Delta_+$ and Θ

These constructions (and some technical details) demonstrate an equivalence between $i\Delta_+$ and Θ_+ where Θ_+ also contains the empty ω-category in addition to the objects of Θ.

We have the following chain of equivalences

$$Disk^{op} \xrightarrow{\simeq} i\mathcal{I}^{op} \xrightarrow{\simeq} i\Delta \xrightarrow{\simeq} \Theta.$$

part II

The remainder of the talk describes two categories, named $t\Delta_+$ and $t\mathcal{I}_+$, constructed from so-called labeled trees, and their reduced counterparts $t\Delta$ and $t\mathcal{I}$ which are also equivalent to the categories *Disk* and Θ (respectively).

The equivalence $t\mathcal{I}_+{}^{op} \simeq t\Delta_+$ is an easy consequence of the duality between ordinals and intervals.

Next we define the concept of a labeled tree.

labeled trees

A <u>labeled tree</u> (A, F) is a tree A with functors $F_n\colon A_n \to \mathcal{C}$

- and so is an object of $Fam(\mathcal{C})$ for each $n \in \mathbb{N}$

A morphism (g, α) of labeled trees is a tree morphism $g\colon A \to B$

- with natural transformations
$$\begin{array}{ccc} A_n & \xrightarrow{g_n} & B_n \\ & {\scriptstyle F_n}\searrow \overset{\alpha_n}{\Rightarrow} \swarrow{\scriptstyle G_n} & \\ & \mathcal{C} & \end{array}$$
and so is a morphism of $Fam_{\Sigma}(\mathcal{C})$ for each $n \in \mathbb{N}$

An op-morphism has natural transformations
$$\begin{array}{ccc} A_n & \xrightarrow{g_n} & B_n \\ & {\scriptstyle F_n}\searrow \overset{\alpha_n}{\Leftarrow} \swarrow{\scriptstyle G_n} & \\ & \mathcal{C} & \end{array}$$

- and so is a morphism of $Fam_{\Pi}(\mathcal{C}^{op})$ for each $n \in \mathbb{N}$

constrained labeled trees

A <u>constrained</u> tree $(A, F, \mathcal{U})$ is a labeled tree (A, F)

- with functors $F_n \colon A_n \to \mathcal{C}$ and $\mathcal{U} \colon \mathcal{C} \to \mathit{Set}$
- such that $A_{n+1} \cong \mathrm{el}(\mathcal{U}F_n)$

A <u>constrained</u> morphism $(f, \alpha) \colon (A, F, \mathcal{U}) \to (B, G, \mathcal{U})$ is a morphism of labeled trees such that the following commutes

$$\begin{array}{ccc} A_{n+1} & \xrightarrow{\cong} & \mathrm{el}(\mathcal{U}F_n) \\ {\scriptstyle f_{n+1}}\downarrow & & \downarrow{\scriptstyle \mathrm{el}(\mathcal{U}\cdot\alpha_n)} \\ B_{m+1} & \xrightarrow{\cong} & \mathrm{el}(\mathcal{U}G_n) \end{array}$$

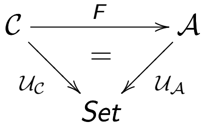
$\mathit{Con}(\mathcal{C}, \mathcal{U})$ is the category of trees

- labeled by functors into $\mathcal{C}$
- constrained by the functor $\mathcal{U} \colon \mathcal{C} \to \mathit{Set}$

the category $Cat/_{Set}$ and functor Con

The 2-category $Cat/_{Set}$ has

- objects: $(\mathcal{C},\mathcal{U})$ with $\mathcal{U}\colon \mathcal{C} \to Set$
- 1-cells: functors $F\colon \mathcal{C} \to \mathcal{A}$ such that



- 2-cells: natural transformations $\alpha\colon F \Rightarrow G$

The 2-functor $Con\colon Cat/_{Set} \to Cat$ has

- $Con(\mathcal{C},\mathcal{U})$ the category of trees labeled in $\mathcal{C}$ constrained by $\mathcal{U}$
- $Con(F)$ given by post-composition; $Con(F)(A,H) = (A, FH)$
- $Con(\alpha)$ given by post-composition; $Con(\gamma)(A,H) = (1_A, \gamma_H)$

Con preserves equivalences and isomorphisms as it is a 2-functor.

using the ordinal/interval duality

From the ordinal/interval duality we have this duality in $Cat/_{Set}$

$$(\mathcal{I}_+, U) \cong (\Delta_+, U(-)^\wedge)$$

and by the functor *Con* have the following duality in *Cat*

$$Con(\mathcal{I}_+, U) \cong Con(\Delta_+, U(-)^\wedge)$$

where U is the ordinary ordinary underlying set functor.

Next we define full subcategories of the latter two categories which are equivalent to the categories *Disk* and Θ (respectively).

cropped trees

Let (A, F) be a tree labeled in $\mathcal{I}_+$ constrained by U

- $x \in A_{n+1}$ is an end element when it corresponds under the isomorphism $A_{n+1} \cong \mathrm{el}(UF_n))$ to an endpoint of $F_n(y)$ for some $y \in A_n$
- (A, F) is cropped when $F_n(x) = [0]$ for every end element x

Let (A, F) be a tree labeled in Δ_+ constrained by $U(\_)^\wedge$

- $x \in A_{n+1}$ is an end element when it corresponds under the isomophism $A_{n+1} \cong \mathrm{el}U(F_n\_)^\wedge$ to an endpoint of $(F_n y)^\wedge$ for some $y \in A_n$
- (A, F) is cropped when $F_n(x) = [-1]$ for every end element x

$t\mathcal{I}_+$ and $t\Delta_+$

Define

- $t\mathcal{I}$ as the full subcategory of $Con(\mathcal{I}_+, U)$ whose objects are cropped trees of positive degree
- $t\Delta$ as the full subcategory of $Con(\Delta_+, U(\_)^\wedge)$ whose objects are cropped trees of positive degree

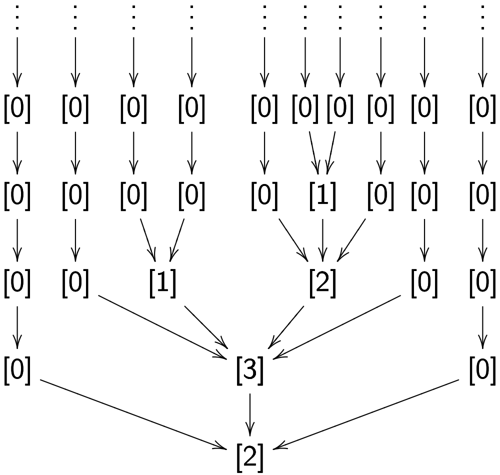
We have an equivalence

$$t\mathcal{I} \simeq t\Delta^{op}$$

as the functors $(\_)^\wedge$ and $(\_)^\vee$ send cropped trees of positive degree to cropped trees of positive degree.

an object of $t\mathcal{I}_+$

e.g. an object of $t\mathcal{I}_+$ of degree 3



conclusion

We have the following equivalences

$$
\begin{array}{ccc}
Disk^{op} & \overset{\simeq}{\dashrightarrow} & \Theta \\
\downarrow \simeq & & \downarrow \simeq \\
i\mathcal{I}^{op} & \overset{\simeq}{\longrightarrow} & i\Delta \\
\downarrow \simeq & & \downarrow \simeq \\
t\mathcal{I}^{op} & \overset{\simeq}{\longrightarrow} & t\Delta
\end{array}
$$

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