The Glueing Construction and Double Categories

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June 23, 2010

What led to double categories

Motivated by "Powerful Functors" (2001 Note)

Street used Benabou's $\mathrm{Cat}/B \simeq \mathrm{Lax}_{\mathrm{N}}(B^{op}, \mathcal{P}\mathit{rof})$, and showed

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Paré suggested double categories (June 2009)

Where I should have used double categories

1978 PhD Thesis (JPAA 23, 1982)

($\star$) $X \subseteq B$ is exponentiable in $\mathrm{Top}/B \Leftrightarrow X$ is locally closed

1980 (Comm Alg 9, 1981)

($\star$) for locales and toposes via Artin-Wraith glueing on

$$\begin{array}{ccc} X_1 & \xrightarrow{f_1} & B_1 \\ {\scriptstyle F}\big\downarrow & \leftarrow & \big\downarrow{\scriptstyle G} \\ X_2 & \xrightarrow[f_2]{} & B_2 \end{array}$$

1998 with Marta Bunge (JPAA 148, 2000)

$P\colon \mathrm{Cat} \to \mathrm{GTop}$ preserves exp (i.e., locally closed) subobjects

2000 (TAC 8, 2001)

exponentiability preservation and ($\star$) for posets

Double Categories

A double category $\mathbb{D}$ is a category object in Cat

$$\mathbb{D}_1 \times_{\mathbb{D}_0} \mathbb{D}_1 \longrightarrow \mathbb{D}_1 \overset{\longrightarrow}{\underset{\longrightarrow}{\longleftarrow}} \mathbb{D}_0$$

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Notation:

Horizontal 1-Cells: $X \xrightarrow{f} B$

Vertical 1-Cells: $X_1 \overset{F}{\nrightarrow} X_2$

2-Cells:
$$\begin{array}{ccc} X_1 & \xrightarrow{f_1} & B_1 \\ {\scriptstyle F}\big\downarrow & \varphi & \big\downarrow{\scriptstyle G} \\ X_2 & \xrightarrow[f_2]{} & B_2 \end{array}$$

Examples

1. $\mathbb{T}$opos: $\mathcal{S}$-toposes $\mathcal{X}$, geom morphisms $\mathcal{X} \xrightarrow{f} \mathcal{B}$, lex $\mathcal{X}_1 \overset{F}{+\!\!\!\to} \mathcal{X}_2$,

$$\begin{array}{ccc} \mathcal{X}_1 & \xrightarrow{f_1} & \mathcal{B}_1 \\ F \downarrow & \Leftarrow & \downarrow G \\ \mathcal{X}_2 & \xrightarrow[f_2]{} & \mathcal{B}_2 \end{array}$$

2. $\mathbb{L}$oc: locales X, locale morphisms $X \xrightarrow{f} B$, lex $X_1 \overset{F}{+\!\!\!\to} X_2$,

$$\begin{array}{ccc} X_1 & \xrightarrow{f_1} & B_1 \\ F \downarrow & \geq & \downarrow G \\ X_2 & \xrightarrow[f_2]{} & B_2 \end{array}$$

3. $\mathbb{T}$op: top spaces X, continuous $X \xrightarrow{f} B$, lex $\mathcal{O}(X_1) \overset{F}{+\!\!\!\to} \mathcal{O}(X_2)$,

$$\begin{array}{ccc} \mathcal{O}(X_1) & \xrightarrow{f_1} & \mathcal{O}(B_1) \\ F \downarrow & \supseteq & \downarrow G \\ \mathcal{O}(X_2) & \xrightarrow[f_2]{} & \mathcal{O}(B_2) \end{array}$$

Examples, cont.

4. $\mathbb{C}$at: categories X, functors $X \xrightarrow{f} B$, profunctors $X_1 \overset{F}{\nrightarrow} X_2$,

$$\begin{array}{ccc} X_1 & \xrightarrow{f_1} & B_1 \\ F \downarrow & \rightarrow & \downarrow G \\ X_2 & \xrightarrow[f_2]{} & B_2 \end{array}$$

5. $\mathbb{P}$os: posets X, order pres $X \xrightarrow{f} B$, order ideals $X_1 \overset{F}{\nrightarrow} X_2$,

$$\begin{array}{ccc} X_1 & \xrightarrow{f_1} & B_1 \\ F \downarrow & \leq & \downarrow G \\ X_2 & \xrightarrow[f_2]{} & B_2 \end{array}$$

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$\mathsf{Lax}_{\mathsf{N}}(B, \mathbb{D})$, for B a small category and $\mathbb{D}$ a double category

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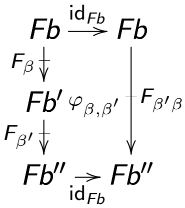

such that $F_{\mathrm{id}_b} = \mathrm{id}_{Fb}$, and coherence

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Note $\mathsf{Lax}_{\mathsf{N}}(\mathbb{1}, \mathbb{D}) = \mathsf{H}\mathbb{D}$, the horizontal category of $\mathbb{D}$

Lax Colimits and Glueing

Suppose B is a small category. Then $\mathbb{D}$ has B-indexed lax colimits or B-glueing if the constant functor

$$H\mathbb{D} \longrightarrow \mathrm{Lax_N}(B, \mathbb{D})$$

has a left adjoint Γ_B. If $\mathbb{D}$ has a (horiz) terminal object T, then so does $\mathrm{Lax_N}(B, \mathbb{D})$ and

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Theorem 1 Suppose $\mathbb{D}$ has finite (horiz) limits and B-glueing. Then $\Gamma_B \colon \mathsf{Lax}_{\mathsf{N}}(B, \mathbb{D}) \longrightarrow \mathsf{H}\mathbb{D}/\Gamma_B T$ has a right adjoint Φ_B.

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Proof. Apply:

Lemma If $\mathcal{C}$ has finite limits, then $\mathcal{C} \xrightarrow{F} \mathcal{D}/D$ has a right adjoint if and only if $\mathcal{C} \xrightarrow{F} \mathcal{D}/D \xrightarrow{\Sigma_D} \mathcal{D}$ does.

□

Description of Φ_B

Assume, in addition, that $\mathbb{D}$ has a (vertical) zero object Z and vertical companions and adjoints. Then there are morphisms

$$\Gamma_{\mathbb{1}} T \xrightarrow{\Gamma_b} \Gamma_B T \qquad\qquad \Gamma_{\mathbb{2}} T \xrightarrow{\Gamma_\beta} \Gamma_B T$$

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for b and $\beta \colon b \to b'$. Then $\Phi_B(X \xrightarrow{p} \Gamma_B T)$ is given by

$$b \mapsto X_b \qquad\qquad \beta \mapsto \left(X_b \xrightarrow{(i_b)_*} X_\beta \xrightarrow{(i_{b'})^*} X_{b'}\right)$$

where X_b and X_β are defined by the pullbacks

$$\begin{array}{ccc} X_b & \xrightarrow{i_b} & X \\ \downarrow & & \downarrow p \\ \Gamma_{\mathbb{1}} T & \xrightarrow[\Gamma_b]{} & \Gamma_B T \end{array} \qquad\qquad \begin{array}{ccc} X_\beta & \xrightarrow{i_b} & X \\ \downarrow & & \downarrow p \\ \Gamma_{\mathbb{2}} T & \xrightarrow[\Gamma_\beta]{} & \Gamma_B T \end{array}$$

Examples

All 5 have zero objects, finite limits, vertical companions/adjoints; $\mathbb{T}$opos has finite lax colimits, and the others have all lax colimits. Thus, Theorem 1 applies.

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2. $\mathbb{L}$oc: $Z = \mathbb{1}$, $T = \Omega$, $\Gamma_B T = \Omega^{\widetilde{B}} = \downarrow\mathrm{Cl}\,\widetilde{B}$, $\widetilde{B}$ the poset $|B|/\sim$
3. $\mathbb{T}$op: $Z = \emptyset$, $T = 1$, $\Gamma_B T = |B|$, "Alexandroff" topology
4. $\mathbb{C}$at: $Z = \emptyset$, $T = \mathbb{1}$, $\Gamma_B T = B^{op}$
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Note Lax colimits in $\mathbb{T}$op are defined as follows:

Given $B \xrightarrow{F} \mathbb{T}\mathrm{op}$, let $\Gamma_B F = \coprod_b Fb$ with $U \subseteq \coprod_b Fb$ is open if $U_b = U \cap Fb$ is open in Fb, $\forall\, b$, and $U_{b'} \subseteq F_\beta(U_b)$, $\forall\, \beta \colon b \to b'$.

The Equivalence

Known: $\mathsf{Lax_N}(\mathbb{2}, \mathbb{D}) \simeq \mathsf{H}\mathbb{D}/\Gamma_{\mathbb{2}} T$ via $\Gamma_{\mathbb{2}}$ and $\Phi_{\mathbb{2}}$, when $\mathbb{D}$ is $\mathbb{T}$opos, $\mathbb{L}$oc (Comm Alg 9), $\mathbb{C}$at, $\mathbb{P}$os (Bénabou).

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Proof. Given $p \colon X \to \mathbb{2}$, let $F0 = p^{-1}0$ and $F1 = p^{-1}1$, with $F \colon \mathcal{O}(F0) \to\!\!\!\!\!\!\mapsto \mathcal{O}(F1)$ given by the "fringe functor"

$$F(U) = (F0 \Rightarrow U) \cap F1 = \mathrm{Int}_X(U \cup F1) \cap F1$$

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Then the continuous bijection $\Gamma_{\mathbb{2}}F \to X$ is an open map, since for U open in $\Gamma_{\mathbb{2}}F$,

$$U = (F0 \Rightarrow U_0) \cap (U_1 \cup F0)$$

and both are open in X.

□

The Equivalence for finite posets.

Applying induction, we get:

Theorem 2 Suppose B is a finite poset, $\mathbb{D}$ has finite limits and B-indexed lax colimits, and $\Gamma_{\mathbb{2}}$ is an equivalence. Then

$$\Gamma_B \colon \mathrm{Lax}_{\mathrm{N}}(B, \mathbb{D}) \xrightarrow{\sim} \mathrm{H}\mathbb{D}/\Gamma_B T$$

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Corollary Suppose B is a finite poset. Then:

1. $\mathrm{Lax}_{\mathrm{N}}(B, \mathbb{T}\mathrm{opos}) \simeq \mathrm{Topos}/\mathcal{S}^{B^{op}}$
2. $\mathrm{Lax}_{\mathrm{N}}(B, \mathbb{L}\mathrm{oc}) \simeq \mathrm{Loc}/{\downarrow}\mathrm{Cl}B$
3. $\mathrm{Lax}_{\mathrm{N}}(B, \mathbb{T}\mathrm{op}) \simeq \mathrm{Top}/B$
4. $\mathrm{Lax}_{\mathrm{N}}(B^{op}, \mathbb{C}\mathrm{at}) \simeq \mathrm{Cat}/B$
5. $\mathrm{Lax}_{\mathrm{N}}(B^{op}, \mathbb{P}\mathrm{os}) \simeq \mathrm{Pos}/B$

Application: Locally Closed Subobjects

Suppose D, D_1 are objects of $\mathbb{D}$. Then D_1 is locally closed in D, if $D = \Gamma_{\mathfrak{Z}} F$ and $D_1 = F1$, for some $F : \mathfrak{Z} \longrightarrow \mathbb{D}$, where $\mathfrak{Z} : 0 \leq 1 \leq 2$.

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Then D_1 arises by glueing $Z \twoheadrightarrow D_1 \twoheadrightarrow Z$, there is a monomorphism $D_1 \longrightarrow D$ in $\mathsf{H}\mathbb{D}$, and $\mathrm{Lax}_{\mathsf{N}}(\mathfrak{Z}, \mathbb{D})/F \simeq \mathsf{H}\mathbb{D}/D$, if $\mathbb{D}$ satisfies (1)–(4):

(1) $\mathbb{D}$ has finite horizontal limits

(2) $\mathbb{D}$ has a zero object Z

(3) $\mathbb{D}$ has $\mathfrak{Z}$-indexed lax colimits

(4) $\Gamma_{\mathfrak{Z}} : \mathrm{Lax}_{\mathsf{N}}(\mathfrak{Z}, \mathbb{D}) \longrightarrow \mathsf{H}\mathbb{D}/\Gamma_{\mathfrak{Z}} T$ is an equivalence

(5) $\mathbb{D}$ has vertical companions and vertical adjoints

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Examples All five examples satisfy (1)–(5).

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Proof. Suppose $i\colon D_1 \longrightarrow D$ is locally closed via F on the right below, and define a right adjoint Π_i to $i^*\colon \mathrm{H}\mathbb{D}/D \longrightarrow \mathrm{H}\mathbb{D}/D_1$ as follows. Given $p_1\colon X_1 \longrightarrow D_1$, consider $X = \Gamma_{\mathbb{3}} G$, where G is on the left below.

$$
\begin{array}{ccccc}
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X_1 & \psi & \\
{\scriptstyle F_2^1(p_1)_*}\big\downarrow & & \\
D_2 & \xrightarrow[\mathrm{id}_{D_0}]{} & D_2
\end{array}
\qquad\qquad
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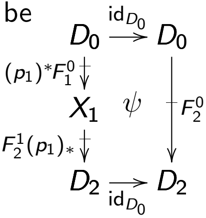

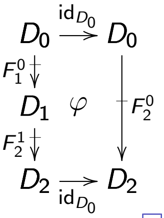

□

Note This gives a construction of exponentials for all 5 examples!