The Notion of Topological Topos

“ A generalization of Johnstone’s Topos”

José Reinaldo Montañez Puentes
Mathematics Department
Universidad Nacional de Colombia

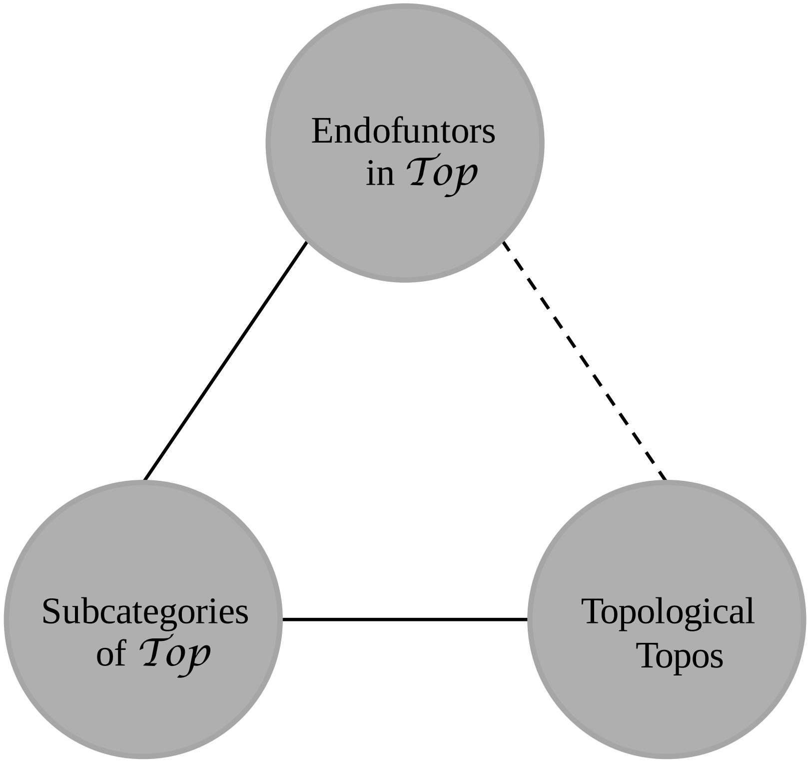
International Category Theory Conference
CT2010
Genova, Italia June 2010

The present work proposes a topos construction method which extend categories of topological spaces

The work comes from the study of a special type of endofuntors defined in *Top*, which we have called Structure Elevators.

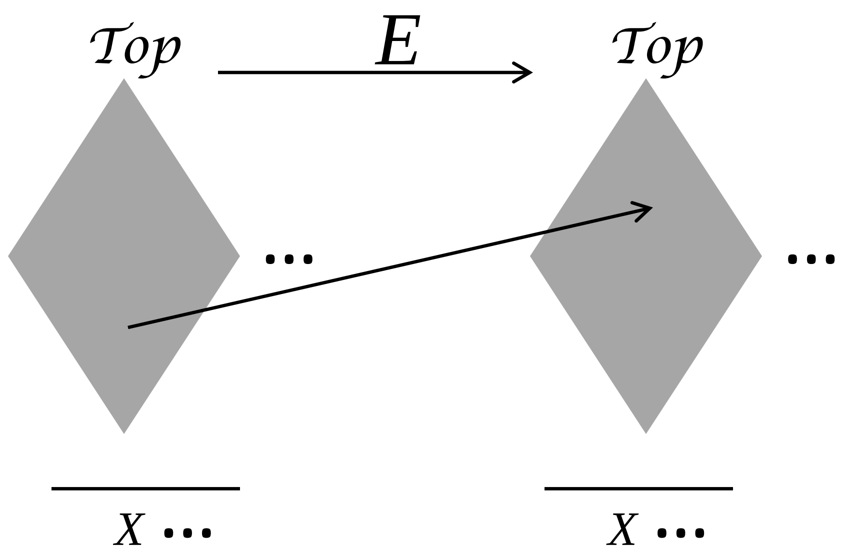
The endofuntors that are idempotent generate topological categories that submerge in topos that we named Topolocal Topos

Towards The Notion of Topological Topos



José Reinaldo Montañez Puentes
jrmontanezp@unal.edu.co

A Structure Elevator is an endofunctor of the topological spaces category that assigns for each topological space another space with the same underlying set but with a finer topology.



Definition

Let $E : \mathcal{T}op \longrightarrow \mathcal{T}op$ be a functor. E is a Structure Elevator if the following conditions are met :

- $O_e \circ E = O_e$
- $X \leq E(X)$ for all object in $\mathcal{T}op$

Topological Categories generated by Idempotent Elevators

$$E : \mathcal{Top} \longrightarrow \mathcal{Top}$$

$$E(\mathcal{Top}) \rightleftarrows \mathcal{Top}$$

1. E($\mathcal{Top}$) is a topological subcategory of $\mathcal{Top}$
2. E($\mathcal{Top}$) is a correflexive subcategory of $\mathcal{Top}$

Representable elevators and representables subcategories

1. Let W be a topological space

$$E_W : \mathcal{T}op \longrightarrow \mathcal{T}op$$
$$X \longrightarrow \varepsilon_f([W,X])$$

2. Examples:

   2.1 Let N_∞ ={1, 1/2, 1/3, . . . , 1/n} ∪ {0} as subspace of the real numbers $\mathbb{R}$. $E_{N\infty}(\mathcal{T}op)$ corresponds to the category of sequential spaces.

   2.2 Let C be a Cantor space. $E_C\,(W)$ corresponds of the category of sequential space.

   2.3Let C be the category of the compact Hausdorff spaces. The elevator $E_C\,(W)$ generates the category of Kelley spaces.

Toposes that extent topological subcategories of $\mathcal{Top}$

Let W be a topological space . Let $M=[W,W]_{\mathcal{Top}}$ be the monoid to endomorphisms of W and $\mathcal{E}_W(\mathcal{Top})$ the topos to associate M-Sets

.

We determinate the functor :

$$\Sigma_W : E_W(\mathcal{Top}) \longrightarrow \mathcal{E}_W(\mathcal{Top})$$

Defined by:

$$\Sigma_W(X) := [W,X]_{\mathcal{Top}} \text{ , } \Sigma_W(f) := \bar{f} \text{ with } \bar{f}(h) = f \circ h$$

Σ_W has adjoin to the left that we will note:

$$L_W : \mathcal{E}_W(\mathcal{Top}) \longrightarrow E_W(\mathcal{Top})$$

Correpresentables coelevators and correpresentables subcategories

1. Let I=[0,1] be a subspace to space of real numbers $\mathbb{R}$. $C_I(\mathcal{Top})$ corresponds to the space complete regular category.

2. Let S be the Sierpinski space. $C_S(\mathcal{Top})$ corresponds to the topological spaces category. The question is whether there is a topological space that generates, by means of an elevator, all the topological spaces.

The Notion of Topological Topos

Let C be a topological subcategory of $\mathcal{Top}$ and $\mathcal{E}$ a topos. We say that $\mathcal{E}$ is a C-topological topos, or simply a topological topos, if $\mathcal{E}$ contains an isomorphic reflective subcategory to C. In such case, the pair $(C, \mathcal{E})$ will be called a Topological Tandem (or simply a Tandem).

Example:

Let W be a topological space. Then the pair $(E_W(\mathcal{Top}), \mathcal{E}_W(\mathcal{Top}))$ is a Tandem.

The M-Set topos formed by the sheaves for the Grothendiek topology "Extensive topology" we have denominated it Extensive Topos.

Extensive topology is generated by ideals that we have denominated Extensive Ideals and that are related to final topologies.

Ideals Extensive
I C [W,W], *W* has topology final for all continuos functions in *I*.

Extensive Topos

Let *W* be a topological space. *Let* $M=[W,W]_{\mathcal{Top}}$ be the monoid of the endomorphisms of *W*.

The topos of sheaves *Sh(W)*

- *Sh(W)* is the subtopos of $E_W(\mathcal{Top})$ formed by the *M*-Sets that are sheaves for the Grotendieck topology determined by the extensive ideals of *M*

Theorem

Let *W* be a topological space. Then,

1. The extensive Topos *Sh(W)* contains as sheaves all of the topological spaces.
2. The extensive Topos *Sh(W)* contains as reflexive subcategory one category isomorphic to the category $E_W(\mathcal{Top})$.
3. The extensive Topos *Sh(W)* is a topological topos.
4. The pair formed by the topological category and the extensive topos $(E_W(\mathcal{Top}),Sh(W))$ is a Tandem.

José Reinaldo Montañez Puentes
jrmontanezp@unal.edu.co

The Notion of Extensive Topos

Definition:

It is said that a topos $\mathcal{E}$ is Extensive if it is equivalent to a topos of the form *Sh(W)* for any topological space *W*.

Examples:

The Jhonstone's topos is an extensive topological-topos. The real numbers corresponds to the convergent successions, in other way $[\mathbb{N}_W, \mathbb{R}]$

Conjecture

Let W be a topological space. The real numbers in the extensive topos generated by *Sh(W)* corresponds to the set of the W continuous functions in the real numbers space $\mathbb{R}$, $[W, \mathbb{R}]_{\mathcal{Top}}$.

Remark

Now, it is to be noted that some topos do not improve with the previous procedure, for example the topos generated by the Sierpinsky space. It is to say, the initially constructed topos coincides with the extensive associated topos.

Towards a general theory of Topological Topos

Remark:

It is easy to notice that the developed theory can be generalized starting from topological constructs C, in other words, Topological functors F to category C in the sets category

Example:

Let C be the category of the Bornological spaces.

Starting from the space of the natural numbers $\mathbb{N}$ with the discrete bornology and imposing the extensive topology on its endomorphism monoid, we determined the Bornological topos that coincides with the proposed by F. W. Lawvere during an on published conference, in Bogota in 1983 and studied subsequently by L. Español, C. Minguez and L. Lamban professors of the Universidad de la Rioja (Spain).

References

J. Adámek., H. Herrlich , G. Strecker , *Abstract and Concrete Categories*, John Wiley and Sons Inc., New York, 1990.

L. Español, C. Minguez, *Cortaduras para I^∞*. Publicación de Margarita Matemática, Universidad de la Rioja, España, (2001) 375-390.

E. Dubuc, J. Zilber, *Infinitesimal and Local Structures for Banach Spaces and its Exponentials in a Topos*, Cahiers de Topologie et géométrie différentille catégoriques, **41** (2000) no. 2 82–100

P.T Johnstone, *On a topological Topos*, Proc. London Math. Soc (3) 38 (1979), 237-271.

L. Lambán, *Construcciones en topos que extienden relaciones entre categorías de espacios topológicos y bornológicos*, Universidad de Zaragoza, Publ. Sem. Mat. García de Galdeano, Ser. II 2 (27), Zaragoza. (1990)

S. Mac Lane, I. Moerdijk, *Sheaves in Geometry and Logic, A first introduction to topos Theory*, Springer-Verlag, New York, 1992.

A. Oostra, *Subcategorías generadas mediante estructuras iniciales*, Lecturas Matemáticas, **16** (1995), 63-72.

A. Oostra, *The Uniformizable Spaces Are Generated by the Real Numbers*, Ann. New York Acad. Sc. **767** (1995), 165-167.

G. Preuss, *Theory of Topological Structures*, D. Reidel Publishing Company, Dordrecht. 1988.

O. Wyler, *Lecture Notes on Topoi and Quasitopoi*, World Scientific, Singapure, 1991.

Representables Elevators and representables subcategories

Remarks:

1. A representable category is determined by the colimits of defined functors of a small category in $\mathcal{Top}$ with the constant value of one of its generator spaces.

2 A representable category is the least correflexive to $\mathcal{Top}$ closed for isomorphism that contain one of its generator spaces