Four problems regarding representable functors

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- **Adjoint, Frobenius, separable functors:**

  If $F : \mathcal{C} \to \mathcal{D}$ is a **left adjoint** of $G : \mathcal{D} \to \mathcal{C}$ we **denote** this by $F \dashv G$. We denote by $\eta : 1_{\mathcal{C}} \to GF$ and $\varepsilon : FG \to 1_{\mathcal{D}}$ the unit and counit of the adjoint pair $F \dashv G$.

Example

The representable functor $Hom_{\mathcal{C}}(C, -) : \mathcal{C} \to \mathcal{S}et$ **has a left adjoint** if and only if C **has arbitrary copowers**, i.e. for any set X there exists the coproduct in $\mathcal{C}$

$$C^{(X)} := \oplus_{x \in X} C_x, \qquad C_x = C, \forall x \in X$$

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- $F : \mathcal{C} \to \mathcal{D}$ is called a **Frobenius functor** if there exists $G : \mathcal{D} \to \mathcal{C}$ such that $F \dashv G \dashv F$.
- Let $F \dashv G$. Then F (resp. G) is a **separable functor** iff η splits (resp. ε cosplits).

Example

Let $\phi : R \to S$ be a morphism of rings. Then $\phi^* : {}_S\mathcal{M} \to {}_R\mathcal{M}$ is Frobenius (resp. separable) iff S/R is a Frobenius (resp. separable) extension of rings in the classical sense.

This basic example was recently generalized to **various theorems** for **Doi-Koppinen modules, Yetter-Drinfel'd modules, entwined modules, modules over corings, etc.**

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Let $(\mathcal{V}, \gamma)$ be a **concrete** category: i.e. $\mathcal{V}$ is a category and $\gamma : \mathcal{V} \to \mathcal{S}et$ a faithful functor.

Typical example: $\mathcal{V}$ is a variety of algebras (like **Gr, Ab, Rings,** k-**Alg**, ${}_R\mathcal{M}$, etc) in the sense of universal algebras and γ is the forgetful functor.

Definition

Let $(\mathcal{V}, \gamma)$ be a concrete category. $F : \mathcal{C} \to \mathcal{V}$ is called a **representable functor** if $\gamma \circ F : \mathcal{C} \to \mathcal{S}et$ is representable in the classical sense, i.e. there exists $C \in \mathcal{C}$, called the **object of representability**, such that

$$\gamma \circ F \cong \mathrm{Hom}_{\mathcal{C}}(C, -)$$

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Four general problems:

Let $\mathcal{C}$ and $(\mathcal{V}, \gamma)$ be given categories.

- Problem A: Describe the category **Rep** $(\mathcal{C}, (\mathcal{V}, \gamma))$ of all representable functors.
- Problem B: Give a necessary and sufficient condition for a given functor $F : \mathcal{C} \to \mathcal{V}$ to be representable (possibly predefining the object of representability).
- Problem C: When is a composition of two representable functors a representable functor?
- Problem D: Give a necessary and sufficient condition for a representable functor $F : \mathcal{C} \to \mathcal{V}$ to be separable or Frobenius.

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Examples

1. Yoneda lema $\Rightarrow$ first answer for Problem A in the **trivial case** $(\mathcal{V}, \gamma) := (\mathcal{S}et, Id_{\mathcal{S}et})$. The functor

$$Y : \mathcal{C}^{\mathrm{op}} \to \textbf{Rep}\,(\mathcal{C}, \mathcal{S}et), \quad Y(C) := \mathrm{Hom}_{\mathcal{C}}(C, -)$$

is an equivalence of categories.

2. **Freyd**'s Theorem (1966): Let $\mathcal{C}$ be a **cocomplete** category, $\mathcal{V}$ a **variety of algebras.** Then $F : \mathcal{C} \to \mathcal{V}$ is representable if and only if F is a right adjoint.

3. (**Freyd + Morita**): Let R and S be rings. Then $F : {}_S\mathcal{M} \to {}_R\mathcal{M}$ is representable if and only if there exists $U \in {}_S\mathcal{M}_R$ such that $F \cong {}_S\mathrm{Hom}(U, -)$.

Consequence: The Problem C has a positive answer for categories of modules. The **tensor product** of bimodules plays the key role as:

$$ {}_R\mathrm{Hom}(V, -) \circ {}_S\mathrm{Hom}(U, -) \cong {}_S\mathrm{Hom}(U \otimes_R V, -) $$

The pioneer of studying Problem A was D. M. Kan in 1958, who described **all representable functors from semigroups to semigroups.**

An excellent book dedicated exclusively to Problem A:
[BH] **G. Bergman, A. Hausknecht**, *Cogroups and Co-rings in categories of associative rings*, AMS, 1996

Example

(**The fundamental theorem [BH]**) Let R be a ring. Then the functor

$$Y : ({}_R\mathcal{M}_R)^{\rm op} \to \textbf{Rep}\,(R\!-\!\textbf{Rings}, \mathcal{A}b), \qquad Y(M) := {}_R{\rm Hom}_R(M, -)$$

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Examples for Problem B

Examples

1. Let $\phi : R \to S$ be a morphism of rings. Then $S \otimes_R - : {}_R\mathcal{M} \to {}_S\mathcal{M}$ is representable **by** $S \in {}_R\mathcal{M}_S$ iff S/R is a Frobenius extension of rings.
2. (T. Brzezinski, 2002) Let R be a ring and C be an R-coring. Then the forgetful functor $F : {}^C_R\mathcal{M} \to {}_R\mathcal{M}$ is representable **by** C iff C is a Frobenius coring.
3. (S. Caenepeel, G.M., S. Zhu, 1997): Let H be a Hopf algebra. Then the forgetful functor $F : {}^H_H\mathcal{Y}D \to {}_H\mathcal{M}$ is representable **by** $H \otimes H$ iff H is finite dimensional and unimodular.

An answer for Problem D is the following:

Theorem

Let $\mathcal{C}$ be an abelian category and $F : \mathcal{C} \to \mathcal{A}b$ be a functor. The following are equivalent:

1. *F is a Frobenius and separable functor;*
2. *$F \cong \mathrm{Hom}_{\mathcal{C}}(C, -)$, for some $C \in \mathcal{C}$ having the following properties:*
   - (i) *C is a small, projective, generator and has all copowers in $\mathcal{C}$;*
   - (ii) *$\mathrm{End}_{\mathcal{C}}(C)$ is Frobenius and separable as a $\mathbf{Z}$-algebra.*

- **Representable functors for corings**
  Let R, S be two rings, C an R-coring and the **cocomplete** categories of comodules

$$ {}^C_R\mathcal{M}, \quad \mathcal{M}^C_R, \quad {}^C_R\mathcal{M}_S, \quad {}^C_R\mathcal{M}^C_R $$

Let $V \in {}^C_R\mathcal{M}_S$. Then we have an adjoint pair of functors

$$ V \otimes_S - : {}_S\mathcal{M} \to {}^C_R\mathcal{M}, \quad {}^C_R\mathrm{Hom}(V, -) : {}^C_R\mathcal{M} \to {}_S\mathcal{M} $$

with $V \otimes_S - \dashv {}^C_R\mathrm{Hom}(V, -)$ (S. Caenepeel, E. De Groot, J. Vercruysse, Trans AMS, 2007).

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The converse of the above proposition also holds.

Theorem

Let R, S be two rings, $\mathcal{C}$ an R-coring and two functors

$$F : {}_S\mathcal{M} \to {}^{\mathcal{C}}_R\mathcal{M}, \quad G : {}^{\mathcal{C}}_R\mathcal{M} \to {}_S\mathcal{M}$$

such that $F \dashv G$, i.e. F is a left adjoint of G.
Then there exists $V \in {}^{\mathcal{C}}_R\mathcal{M}_S$, unique up to an isomorphism in ${}^{\mathcal{C}}_R\mathcal{M}_S$, such that

$$F \cong V \otimes_S - \qquad G \cong {}^{\mathcal{C}}_R\mathrm{Hom}(V, -)$$

Let $\mathbf{Rep}\,({}^{C}_{R}\mathcal{M}, {}_{S}\mathcal{M})$ be the category of all representable functors ${}^{C}_{R}\mathcal{M} \to {}_{S}\mathcal{M}$.

The above theorem + Freyd's theorem + a Yoneda type embedding for comodules over corings give:

Corollary

Let R, S be rings, C an R-coring. Then the functor

$$Y : ({}^{C}_{R}\mathcal{M}_{S})^{\mathrm{op}} \to \mathbf{Rep}\,({}^{C}_{R}\mathcal{M}, {}_{S}\mathcal{M}), \quad Y(V) := {}^{C}_{R}\mathrm{Hom}(V, -)$$

is an equivalence of categories.

Two questions:

- Describe the category **Rep** $({}_S\mathcal{M}, {}^C_R\mathcal{M})$ (or more general **Rep** $({}^D_S\mathcal{M}, {}^C_R\mathcal{M})$, for a S-coring D) of all representable functors ${}_S\mathcal{M} \to {}^C_R\mathcal{M}$.
- Describe all **right adjoint** functors $G : {}_S\mathcal{M} \to {}^C_R\mathcal{M}$.

  Partial answer (T. Brzezinski, R. Wisbauer): if $C \in \mathcal{M}_R$ is flat, G **preserves all colimits** and its left adjoint F **preserves kernels**, then

$$G \cong V \otimes_S -, \quad F \cong M \square_C -$$

  for some $V \in {}^C_R\mathcal{M}_S$, $M \in {}_S\mathcal{M}^C_R$.

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Let $U \in {}_S\mathcal{M}_R$ and the induction functor $U \otimes_R - : {}^C_R\mathcal{M} \to {}_S\mathcal{M}$

• When is $U \otimes_R - : {}^C_R\mathcal{M} \to {}_S\mathcal{M}$ a representable functor?

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Example

Let $U := S := R$. Then $R \otimes_R - \cong F$, where $F : {}^C_R\mathcal{M} \to {}_R\mathcal{M}$ is the forgetful functor. We have the adjoint pairs:

$$F = R \otimes_R - \dashv C \otimes_R - \dashv {}^C_R\mathrm{Hom}(C, -)$$

Thus, $F \cong R \otimes_R -$ is a representable functor **by** C
$\Longleftrightarrow$ F is a Frobenius functor
$\Longleftrightarrow$ C is a Frobenius coring (T. Brzezinski).

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Theorem

Let C be a R-coring and $U \in {}_S\mathcal{M}_R$. TFAE:

1. $U \otimes_R - : {}_R^C\mathcal{M} \to {}_S\mathcal{M}$ *is an equivalence of categories;*
2. *There exists* $V \in {}_R^C\mathcal{M}_S$ *such that:*
   - (i) $U \otimes_R V \cong S$, *isomorphism in* ${}_S\mathcal{M}_S$.
   - (ii) $V \otimes_S U \otimes_R C \cong C$, *isomorphism in* ${}_R^C\mathcal{M}_R^C$;
3. *There exists a triple* (V, p, h), *where* $V \in {}_R^C\mathcal{M}_S$, $p \in {}_R^C\mathrm{Hom}_R(C, V \otimes_S U)$, $h \in {}_R\mathrm{Hom}_R(V \otimes_S U \otimes_R C, R)$ *such that:*
   - (i) $U \otimes_R V \cong S$, *isomorphism in* ${}_S\mathcal{M}_S$;
   - (ii) $v_{<-1>}h(v_{<0>} \otimes_S u \otimes_R c) = h(v \otimes_S u \otimes_R c_{(1)})c_{(2)}$;
   - (iii) $h(p(c_{(1)}) \otimes_R c_{(2)}) = \varepsilon(c)$;
   - (iv) $h(v \otimes_S u \otimes_R c_{(1)})p(c_{(2)}) = v \otimes_S u\varepsilon(c)$

   for all $v \in V$, $u \in U$, $c \in C$.

Definition

Let R, S be two rings, C an R-coring, $U \in {}_S\mathcal{M}_R$ and $V \in {}^C_R\mathcal{M}_S$.

A pair (e, h), where $e = \sum e^1 \otimes e^2 \in \big(U \otimes_R V\big)^S$,
$h \in {}_R\mathrm{Hom}_R(V \otimes_S U \otimes_R C, R)$, such that

$$v_{<-1>}h(v_{<0>} \otimes_S u \otimes_R c) = h(v \otimes_S u \otimes_R c_{(1)})c_{(2)}$$

$$\sum e^1\, h(e^2 \otimes_S u \otimes_R c) = u\varepsilon(c)$$

$$\sum h(v \otimes_S e^1 \otimes_R e^2_{<-1>})e^2_{<0>} = v$$

for all $v \in V$, $u \in U$, $c \in C$ is called a **comodule dual basis of first kind** for (U, V).

Example

Let $C := R$, $U \in {}_S\mathcal{M}_R$, $V := U^* = \mathrm{Hom}_R(U, R) \in {}_R\mathcal{M}_S$ and h the evaluation map

$$h := ev_U : U^* \otimes_S U \to R, \quad u^* \otimes_S u \mapsto < u^*, u >$$

There exists $e = \sum_i u_i \otimes_R u_i^* \in (U \otimes_R U^*)^S$ such that $(e, h = ev_U)$ is a 'comodule' dual basis of first kind for (U, U^*) iff $\{u_i, u_i^*\}$ is a dual basis for $U \in \mathcal{M}_R$ iff U is finitely generated projective as a right R module.

Theorem

Let C be an R-coring and $U \in {}_S\mathcal{M}_R$. TFAE:

1. $U \otimes_R - : {}^C_R\mathcal{M} \to {}_S\mathcal{M}$ *is a representable functor;*
2. *There exists* $V \in {}^C_R\mathcal{M}_S$ *such that* $V \otimes_S - \dashv U \otimes_R -$;
3. *There exists* (V, e, h), *where* $V \in {}^C_R\mathcal{M}_S$ *and* (e, h) *is a comodule dual basis of first kind for* (U, V).

In this case $U \otimes_R - \cong {}^C_R\mathrm{Hom}(V, -)$ *and* $V \in {}_R\mathcal{M}$ *is f.g. projective.*

Theorem

Let R, S be two rings, C an R-coring, $U \in {}_S\mathcal{M}_R$ and $V \in {}^C_R\mathcal{M}_S$. TFAE:

1. *Then the induction functor $U \otimes_R - : {}^C_R\mathcal{M} \to {}_S\mathcal{M}$ is a left adjoint of $V \otimes_S - : {}_S\mathcal{M} \to {}^C_R\mathcal{M}$;*
2. *There exists a pair (p, E), where $p \in {}^C_R\mathrm{Hom}_R(C, V \otimes_S U)$, $E \in {}_S\mathrm{Hom}_S(U \otimes_R V, S)$, such that*

$$(Id_V \otimes_S E) \circ (p \otimes_R Id_V) \circ \rho_V = Id_V$$

$$(E \otimes_S Id_U) \circ (Id_U \otimes_R p) = Id_U \otimes_R \varepsilon_C$$

A pair of maps (p, E) as above is called a ***comodule dual basis of second kind*** *for (U, V).*

Example

Let $C := R$, $V \in {}_R\mathcal{M}_S$, $U := V^*$ and the evaluation map

$$E : V^* \otimes_R V \to S, \quad E(v^* \otimes_R v) = < v^*, v >$$

There exists $p \in {}_R\mathrm{Hom}_R(R, V \otimes_S V^*)$ such that (p, E) is a 'comodule' dual basis of second kind for (V^*, V) iff V is f.g. projective as a right S-module.

Corollary

Let R, S be two rings, C an R-coring, $U \in {}_S\mathcal{M}_R$ and $V \in {}^{C}_{R}\mathcal{M}_S$. TFAE:

1. *The pair of functors $(V \otimes_S -, U \otimes_R -)$ is a Frobenius pair;*
2. *There exists $((e, h), (p, E))$ such that (e, h) (resp. (p, E)) is a comodule dual basis of first kind (resp. second kind) for (U, V).*

Corollary

Let R, S be two rings, C an R-coring, $U \in {}_S\mathcal{M}_R$ and $V \in {}^C_R\mathcal{M}_S$. Assume that there exists (e, h) a comodule dual basis of first kind for (U, V). Then:

1. *The induction functor $V \otimes_S - : {}_S\mathcal{M} \to {}^C_R\mathcal{M}$ is separable if and only if there exists $E \in {}_S\mathrm{Hom}_S(U \otimes_R V, S)$ such that $E(e) = 1$.*
2. *The induction functor $U \otimes_R - : {}^C_R\mathcal{M} \to {}_S\mathcal{M}$ is separable if and only if there exists $p \in {}^C_R\mathrm{Hom}_R(C, V \otimes_S U)$ such that:*

$$h(p(c_{(1)}) \otimes_R c_{(2)}) = \varepsilon(c)$$

for all $c \in C$.

The dual version of it:

Corollary

Let R, S be two rings, C an R-coring, $U \in {}_S\mathcal{M}_R$ and $V \in {}^C_R\mathcal{M}_S$. Assume that there exists (p, E) a comodule dual basis of second kind for (U, V). Then:

1. *The induction functor $V \otimes_S - : {}_S\mathcal{M} \to {}^C_R\mathcal{M}$ is separable if and only if there exists an element $e \in (U \otimes_R V)^S$ such that $E(e) = 1$.*
2. *The induction functor $U \otimes_R - : {}^C_R\mathcal{M} \to {}_S\mathcal{M}$ is separable if and only if there exists $h \in {}_R\mathrm{Hom}_R(V \otimes_S U \otimes_R C, R)$ s.t.:*

$$v_{<-1>}h(v_{<0>} \otimes_S u \otimes_R c) = h(v \otimes_S u \otimes_R c_{(1)})c_{(2)}$$

$$h\big(p(c_{(1)}) \otimes_R c_{(2)}\big) = \varepsilon(c)$$

for all $v \in V$, $u \in U$, $c \in C$.

Remark

The separability of the induction functor is still an open problem even for the category of modules (i.e. for the trivial coring $C := R$): S. Caenepeel and L. Kadison (K-Theory, 2001) solved the problem only for ***finitely generated and projective modules****.*

Thank You!

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