The Hopf algebra of Möbius intervals

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The idea of generating function

‘Since Laplace discovered the remarkable correspondence between set theoretic operations on formal power series, and put it to use with great success to solve a variety of combinatorial problems, generating functions [...] have become an essential probabilistic and combinatorial technique. [...] in order to extend the theory beyond its present reaches and develop new kinds of algebras of generating functions [...] it seems necessary to abandon the notion of group algebra [...] and rely instead on an altogether different approach.’

Doubilet, Rota and Stanley.
On the foundations of
combinatorial theory (VI).
1972.

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FUNDAMENTAL TOOL: Characterization of invertible elements of incidence algebras.

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Let $\mathcal{C}_1$ be the set of maps in $\mathcal{C}$ and A a ring. Define

$A\mathcal{C} = (\{\mathcal{C}_1 \to A\}, +, 0, *, \delta)$ where

$$(\alpha * \beta)f = \sum_{f'f''=f} (\alpha f')(\beta f'')$$

and

$$\delta f = \begin{cases} 1 & f = id \\ 0 & f \neq id \end{cases}$$

Examples

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What categories induce ‘good’ incidence algebras?

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Proposition 4. *$\mathcal{C}$ with finite decompositions of degree 2 is Möbius iff*

1. *Identities are 'indecomposable' and*
2. *$gh = g$ implies $h = id$.*

Characterization in terms of localness

Definition 11. A category is called pre-Möbius if it has finite decompositions of degree 2 and its identities are indecomposable.

If $\mathcal{C}$ is a pre-Möbius category, the functor $|\mathcal{C}| \to \mathcal{C}$ induces, for each ring A, a (restriction) algebra map $A\mathcal{C} \to A|\mathcal{C}|$.

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Theorem 2 (Leroux)**.** *For a pre-Möbius category $\mathcal{C}$, t.f.a.e.:*

1. *$\mathcal{C}$ is Möbius*
2. *$A\mathcal{C} \to A|\mathcal{C}|$ is local for each ring A.*

Proof. For α in $A\mathcal{C}$ s.t. $\alpha(id)$ invertible. Define α^{-1} using a recursive formula for $\alpha^{-1}f$, for each map f in $\mathcal{C}$. □

Examples

Recall the statement:

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□

Corollary 1. *For any ring A,*

1. *($A(\mathbb{N}, +, 0) \to A$ local) A power series in $A[[x]]$ is invertible iff its constant term is.*
2. *($A\omega \to A\mathbb{N}$ local) An upper triangular matrix is invertible iff its diagonal elements are invertible.*

The Möbius function μ

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Definition 13. The Riemann function ζ of $A\mathcal{C}$ is the element defined by $\zeta f = 1 \in A$ for every f in $\mathcal{C}$.

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Example 27.

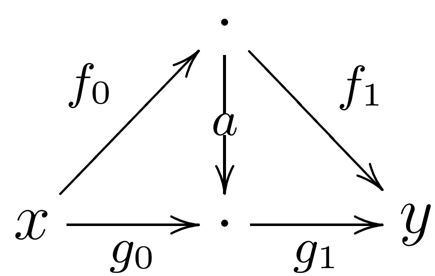
1. (In $A[[x]]$) $\zeta(x) = \sum_n x^n$ and $\mu(x) = 1 - x$.
   (So, $\zeta = \frac{1}{1-x}$ is fair.)
2. (Dirichlet) $\zeta(s) = \sum_n \frac{1}{n^s}$ and $\mu(s) = \sum_n \frac{\mu' n}{n^s}$, where μ' is the classical Möbius function.
3. (For $\mathcal{C}$ a finite Boolean algebra) $\mu(T \subseteq S) = (-1)^{|S-T|}$

Möbius intervals

Intervals

Let $f : x \to y$ be a map in a category $\mathcal{C}$.

Definition 16. The interval $\mathbf{I}f$ is the category constructed as follows. An object of $\mathbf{I}f$ is a pair (f_0, f_1) of maps of $\mathcal{C}$ for which $f = f_1 f_0$ and a map $a : (f_0, f_1) \to (g_0, g_1)$ in $\mathbf{I}f$ is any map of $\mathcal{C}$ such that



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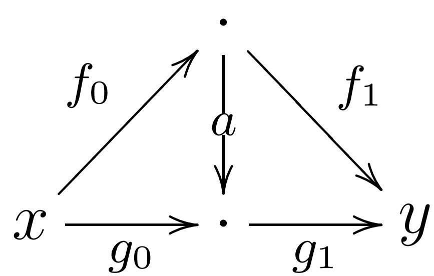

Example 29.

1. (if $x \leq y$ in a poset $\mathcal{C}$) $\mathbf{I}(x \leq y) = \{u \mid x \leq u \leq y\}$ with poset structure inherited from $\mathcal{C}$.
2. (if $\mathcal{C} = (\mathbb{N}, +, 0)$) $\mathbf{I}n$ = totally ordered set with $n + 1$ elements.

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Theorem 5. *For a small category $\mathcal{C}$ t.f.a.e:*

1. *$\mathcal{C}$ is Möbius*
2. *all intervals of $\mathcal{C}$ are finite one-way.*

Proof. Intervals of Möbius categories are finite and Möbius.
A finite category is Möbius iff it is one way. □

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Let $\mathbf{sM\ddot{o}I}$ be the category of Möbius intervals and strict functors between them.

Consider the category $[\mathbf{sM\ddot{o}I}, \mathbf{Set}_f]$.

Let $\zeta : \mathbf{sM\ddot{o}I} \to \mathbf{Set}_f$ be the terminal object of $[\mathbf{sM\ddot{o}I}, \mathbf{Set}_f]$. It assigns the terminal set 1 to each Möbius interval.

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Let $\delta : \mathbf{sM\ddot{o}I} \to \mathbf{Set}_f$ map the trivial interval 1 to the final set 1 and every other interval to the empty set 0. (This can be done because 1 is disconnected in $\mathbf{sM\ddot{o}I}$.)

The ‘dual’ of sMöI

sMöI as a ‘comonoid’

Let $\Delta : \mathbf{sM\ddot{o}I} \to \mathrm{Fam}(\mathbf{sM\ddot{o}I} \times \mathbf{sM\ddot{o}I})$ be defined by

$$\Delta\mathcal{C} = \sum_{x \in \mathcal{C}} (x/\mathcal{C}, \mathcal{C}/x)$$

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The ‘comonoid’ $(\mathbf{sM\ddot{o}I}, \Delta, \delta)$ induces a ‘distributive’ monoidal structure $([\mathbf{sM\ddot{o}I}, \mathbf{Set}_f], *, \delta)$.

For α, β in $[\mathbf{sM\ddot{o}I}, \mathbf{Set}_f]$,

$$(\alpha * \beta)\mathcal{C} = \sum_{x \in \mathcal{C}} \alpha(x/\mathcal{C}) \times \beta(\mathcal{C}/x)$$

for $\mathcal{C}$ in $\mathbf{sM\ddot{o}I}$.

Möbius ‘inversion’

$\Phi_+ : \mathbf{sM\ddot{o}I} \to \mathbf{Set}_f$ maps each $\mathcal{C}$ in $\mathbf{sM\ddot{o}I}$ to the set $\Phi_+\mathcal{C}$ of even-length decompositions of the unique map $0 \to 1$ in $\mathcal{C}$. (If $F : \mathcal{C} \to \mathcal{D}$ is a strict functor and $(f_1, \ldots, f_n)$ is in $\Phi_+\mathcal{C}$ then $(Ff_1, \ldots, Ff_n)$ is in $\Phi_+\mathcal{D}$.)

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Proposition 7. *There is an iso* $\delta + \zeta * \Phi_- \longrightarrow \zeta * \Phi_+$ *in the monoidal extensive category* $[\mathbf{sM\ddot{o}I}, \mathbf{Set}_f]$.

Proof. Very explicit (and non-recursive) definition involving choping decompositions. □

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Intuition: The iso $\delta + \zeta * \Phi_- \cong \zeta * \Phi_+$ is the 'combinatorial reason' why $\mu = \Phi_+ - \Phi_-$ in incidence algebras.

Burnside rigs and algebras

Algebras of iso classes

If $(\mathcal{C}, \otimes, u)$ is an essentially small monoidal category the set of iso-classes of $\mathcal{C}$ inherits a monoid structure with multiplication $[C] \cdot [C'] = [C \otimes C']$ and unit $[u]$.

This monoid will be called the Burnside monoid associated to the monoidal category and will be denoted by $\mathfrak{B}(\mathcal{C}, \otimes, u)$.

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Definition 31. A rig A is a set equipped with two commutative monoid structres $(A, +, 0)$ and $(A, \cdot, 1)$ and such that $x(a+b) = xa + xb$ and $0a = 0$.

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If $\mathcal{C}$ is a prextensive category then $\mathfrak{B}\mathcal{C}$ is the Burnside rig of $\mathcal{C}$ (Schanuel, Como, 1990).

If $\mathcal{C}$ is extensive and monoidal 'distributive' then $\mathfrak{B}\mathcal{C}$ is a non-commutative rig.

The Burnside monoid of $\mathrm{sM\ddot{o}I}$

Let $\mathcal{I} = \mathfrak{B}(\mathbf{sM\ddot{o}I})$ be the monoid of isomorphism classes in $\mathbf{sM\ddot{o}I}$ where $[\mathcal{C}] \cdot [\mathcal{D}] = [\mathcal{C} \times \mathcal{D}]$. ($\times$ denotes the product as categories. It is NOT categorical product in $\mathbf{sM\ddot{o}I}$.)

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Let $\mathbb{N}[\mathcal{I}]$ be the associated monoid-rig.

The assignment $[\mathcal{C}] \mapsto \sum_{x \in \mathcal{C}} [x/\mathcal{C}] \otimes [\mathcal{C}/x]$ induces a 'linear' $\Delta : \mathbb{N}[\mathcal{I}] \longrightarrow \mathbb{N}[\mathcal{I}] \otimes \mathbb{N}[\mathcal{I}]$.
Together with the obvious $\delta : \mathbb{N}[\mathcal{I}] \longrightarrow \mathbb{N}$, they determine a comonoid structure $(\mathbb{N}[\mathcal{I}], \Delta, \delta)$ in $\mathbb{N}$-Mod.

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For each rig A, the associated convolution algebra will be denoted by $\mathrm{Hom}(\mathbb{N}[\mathcal{I}], A)$.

Notice: There are obvious $\zeta, \Phi_+, \Phi_- \in \mathrm{Hom}(\mathbb{N}[\mathcal{I}], A)$.

Master Möbius inversion

Proposition 9. *The assignment*

$$(F : \mathbf{sM\ddot{o}I} \to \mathbf{Set}_f) \mapsto (\mathfrak{B}F : \mathcal{I} \to \mathbb{N})$$

induces an algebra map

$$\mathfrak{B}[\mathbf{sM\ddot{o}I}, \mathbf{Set}_f] \to \mathbf{Set}(\mathcal{I}, \mathbb{N}) \cong \mathrm{Hom}(\mathbb{N}[\mathcal{I}], \mathbb{N})$$

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Corollary 9. $\delta + \zeta * \Phi_- = \zeta * \Phi_+$ *holds in* $\mathrm{Hom}(\mathbb{N}[\mathcal{I}], \mathbb{N})$*.*

The unique $\mathbb{N} \to \mathbb{Z}$ induces an algebra map $\mathrm{Hom}(\mathbb{N}[\mathcal{I}], \mathbb{N}) \to \mathrm{Hom}(\mathbb{N}[\mathcal{I}], \mathbb{Z})$.

Corollary 10. $\zeta \in \mathrm{Hom}(\mathbb{N}[\mathcal{I}], \mathbb{Z})$ *is invertible and its inverse* μ *equals* $\Phi_+ - \Phi_-$*.*

The ‘master’ algebra

Incidence algebras and convolution

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$$f \mapsto \sum_{(f_0, f_1) \in \mathbf{I} f} f_1 \otimes f_0$$

induces a linear $\Delta : \mathbb{N}[\mathcal{C}_1] \to \mathbb{N}[\mathcal{C}_1] \otimes \mathbb{N}[\mathcal{C}_1]$. Together with the obvious $\delta : \mathbb{N}[\mathcal{C}_1] \to \mathbb{N}$ they induce a comonoid $(\mathbb{N}[\mathcal{C}_1], \Delta, \delta)$.

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Lemma 3. *If A is a ring, the convolution algebra $\mathrm{Hom}(\mathbb{N}[\mathcal{C}_1], A)$ coincides with the incidence algebra $A\mathcal{C}$.*

The Master Algebra

Lemma 4. *The assignment* $f \mapsto \mathbf{I}f$ *extends to a comonoid morphism* $\mathbf{I} : (\mathbb{N}[\mathcal{C}_1], \Delta, \delta) \to (\mathbb{N}[\mathcal{I}], \Delta, \delta)$.

The Master Algebra

Lemma 5. *The assignment $f \mapsto \mathbf{I}f$ extends to a comonoid morphism $\mathbf{I} : (\mathbb{N}[\mathcal{C}_1], \Delta, \delta) \to (\mathbb{N}[\mathcal{I}], \Delta, \delta)$.*

Theorem 7 (Lawvere)**.** *For every Möbius category $\mathcal{C}$ and every ring A there is a canonical algebra map $\mathrm{Hom}(\mathbb{N}[\mathcal{I}], \mathbb{Z}) \to A\mathcal{C}$ preserving ζ, Φ_+, Φ_-.*

Proof. The comonoid map $\mathbf{I} : (\mathcal{C}_1, \Delta, \delta) \to (\mathbb{N}[\mathcal{I}], \Delta, \delta)$ induces $\mathrm{Hom}(\mathbb{N}[\mathcal{I}], \mathbb{Z}) \to \mathbb{Z}\mathcal{C}$. Then postcompose with the algebra map $\mathbb{Z}\mathcal{C} \to A\mathcal{C}$ induced by $\mathbb{Z} \to A$. □

The Master Algebra

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So get the Möbius algebra where all ‘natural’ elements of incidence algebras live.

The Master Algebra

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Proof. The comonoid map $\mathbf{I} : (\mathcal{C}_1, \Delta, \delta) \to (\mathbb{N}[\mathcal{I}], \Delta, \delta)$ induces $\mathrm{Hom}(\mathbb{N}[\mathcal{I}], \mathbb{Z}) \to \mathbb{Z}\mathcal{C}$. Then postcompose with the algebra map $\mathbb{Z}\mathcal{C} \to A\mathcal{C}$ induced by $\mathbb{Z} \to A$. □

So get the Möbius algebra where all 'natural' elements of incidence algebras live.

Corollary 14 (Leroux et. al.)**.** *In* $A\mathcal{C}$, $\mu = \Phi_+ - \Phi_-$.

The ‘Hopf rig’ of Möbius intervals

The monoid rig $\mathbb{N}[\mathcal{I}]$ together with the comonoid structure $(\mathbb{N}[\mathcal{I}], \Delta, \delta)$ is a bialgebra. (So the convolution algebra $\mathrm{Hom}(\mathbb{N}[\mathcal{I}], \mathbb{N}[\mathcal{I}])$ makes sense.)

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Proposition 13. *There are linear maps $S_+, S_- : \mathbb{N}[\mathcal{I}] \longrightarrow \mathbb{N}[\mathcal{I}]$ such that*

$$\delta + id * S_- = id * S_+ \qquad \delta + S_- * id = S_+ * id$$

where $id \in \mathrm{Hom}(\mathbb{N}[\mathcal{I}], \mathbb{N}[\mathcal{I}])$.

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(Think of (S_+, S_-) as an ‘antipode’.)

Consider the monoid-ring $\mathbb{Z}[\mathcal{I}]$ and the extensions $S_+, S_- : \mathbb{Z}[\mathcal{I}] \to \mathbb{Z}[\mathcal{I}]$.

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Consider the monoid-ring $\mathbb{Z}[\mathcal{I}]$ and the extensions $S_+, S_- : \mathbb{Z}[\mathcal{I}] \longrightarrow \mathbb{Z}[\mathcal{I}]$.

Corollary 18 (Lawvere). $\mathbb{Z}[\mathcal{I}]$ *is a Hopf algebra.*

Proof. $S = S_+ - S_-$ is an antipode. □

The induced algebraic groups

Let $\mathcal{C}$ be a Möbius category.

Taking subgroup $G_{\mathcal{C}}A \to A\mathcal{C}$ of invertible elements induces a 'particular' functor $G_{\mathcal{C}} : \mathbf{Ring} \to \mathbf{Grp}$.

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And there are maps $G \to G_{\mathcal{C}}$

$$\begin{array}{ccc} G & \longrightarrow & G_{\mathcal{C}} \\ \downarrow & & \downarrow \\ \mathrm{Hom}(\mathbb{N}[\mathcal{I}], \_) & \longrightarrow & \mathrm{Hom}(\mathbb{N}[\mathcal{C}_1], \_) \end{array}$$

from the ‘general’ algebraic group to each ‘particular’ one.

Unique lifting of factorizations

Unique Lifting of Factorizations

Let $F : \mathcal{C} \to \mathcal{D}$ be a functor.

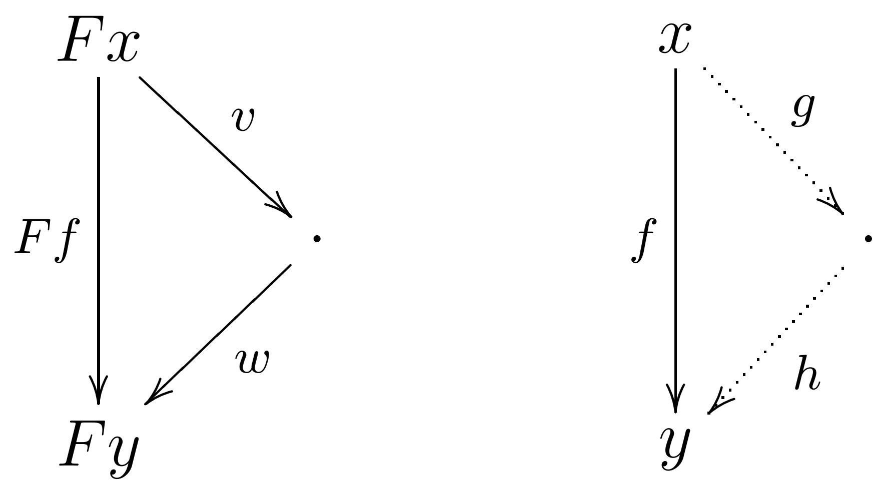
Any $f : x \to y$ in $\mathcal{C}$ induces a functor $F_f : \mathbf{I}f \to \mathbf{I}(Ff)$ s.t.

$$\begin{array}{ccc} \mathbf{I}f & \xrightarrow{F_f} & \mathbf{I}(Ff) \\ \downarrow & & \downarrow \\ \mathcal{C} & \xrightarrow[F]{} & \mathcal{D} \end{array}$$

commutes.

ULF functors...

Lemma 8. *$F_f : \mathbf{I}f \to \mathbf{I}(Ff)$ is an iso for every f in $\mathcal{C}$ iff for every map $f : x \to y$ in $\mathcal{C}$, if $Ff = wv$ as on the left below*



then $\exists!$ pair g, h of maps in $\mathcal{C}$ such that $hg = f$ as on the right above and such that $Fg = v$ and $Fh = w$.

A functor $F : \mathcal{C} \to \mathcal{D}$ satisfying these equivalent conditions is called ULF.

... and maps of algebras

Theorem 10. *For every functor* $F : \mathcal{C} \longrightarrow \mathcal{D}$ *between Möbius categories the following are equivalent:*

1. F *satisfies the ULF condition,*
2. $F : \mathbb{N}[\mathcal{C}_1] \longrightarrow \mathbb{N}[\mathcal{D}_1]$ *is a comonoid morphism,*
3. *for every rig* A, $F_A : A\mathcal{D} \longrightarrow A\mathcal{C}$ *is an algebra map.*

Example 30.

... and maps of algebras

Theorem 11. *For every functor $F : \mathcal{C} \longrightarrow \mathcal{D}$ between Möbius categories the following are equivalent:*

1. *F satisfies the ULF condition,*
2. *$F : \mathbb{N}[\mathcal{C}_1] \longrightarrow \mathbb{N}[\mathcal{D}_1]$ is a comonoid morphism,*
3. *for every rig A, $F_A : A\mathcal{D} \longrightarrow A\mathcal{C}$ is an algebra map.*

Example 31.

1. The inclusion $|\mathcal{C}| \longrightarrow \mathcal{C}$ induces restrictions $A\mathcal{C} \longrightarrow A|\mathcal{C}|$.

... and maps of algebras

Theorem 12. *For every functor $F : \mathcal{C} \to \mathcal{D}$ between Möbius categories the following are equivalent:*

1. *F satisfies the ULF condition,*
2. *$F : \mathbb{N}[\mathcal{C}_1] \to \mathbb{N}[\mathcal{D}_1]$ is a comonoid morphism,*
3. *for every rig A, $F_A : A\mathcal{D} \to A\mathcal{C}$ is an algebra map.*

Example 32.

1. The inclusion $|\mathcal{C}| \to \mathcal{C}$ induces restrictions $A\mathcal{C} \to A|\mathcal{C}|$.
2. $\omega \to \mathbb{N}$ mapping $m \leq n$ to $n - m$ induces inclusions $A\mathbb{N} \to A\omega$ of power series rings into algebras of triangular matrices.

... and maps of algebras

Theorem 13. *For every functor $F : \mathcal{C} \to \mathcal{D}$ between Möbius categories the following are equivalent:*

1. *F satisfies the ULF condition,*
2. *$F : \mathbb{N}[\mathcal{C}_1] \to \mathbb{N}[\mathcal{D}_1]$ is a comonoid morphism,*
3. *for every rig A, $F_A : A\mathcal{D} \to A\mathcal{C}$ is an algebra map.*

Example 33.

1. The inclusion $|\mathcal{C}| \to \mathcal{C}$ induces restrictions $A\mathcal{C} \to A|\mathcal{C}|$.
2. $\omega \to \mathbb{N}$ mapping $m \leq n$ to $n - m$ induces inclusions $A\mathbb{N} \to A\omega$ of power series rings into algebras of triangular matrices.
3. $(\mathbb{N}^*, |) \to \mathbb{N}^*$ mapping $m \mid n$ to $\frac{n}{m}$ induces inclusions $A\mathbb{N}^* \to A(\mathbb{N}^*, |)$ of rings of Dirichlet series into incidence algebras of posets.

The end

$$
\begin{array}{ccccc}
\mathbf{M\ddot{o}I} & \longrightarrow & \mathbf{M\ddot{o}} & \longrightarrow & \mathbf{Cat} \\
\uparrow & & \uparrow & & \\
\mathbf{sM\ddot{o}I} & & \mathbf{ulfM\ddot{o}} & &
\end{array}
$$