Pseudomonads — No iteration version

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Manes' exercise on monads

Exercise 1.3.12 A monad $\mathbb{S}$ on a category **C** is equivalent to:

- A *function* $|S| : |\mathbf{C}| \to |\mathbf{C}|$;
- for every $A \in \mathbf{C}$, an arrow $\eta A : A \to SA$;
- for every morphism $f : B \to SA$ in **C**, an $\mathbb{S}$-extension $f^{\mathbb{S}} : SB \to SA$.

Subject to the axioms:

- for every A in **C**,

$$(\eta A)^{\mathbb{S}} = 1_{SA};$$

- for every $f : B \to SA$ in **C** and $g : C \to SB$, the diagrams

$$\begin{array}{ccc} B & \xrightarrow{\eta B} & SB \\ & {\scriptstyle f}\searrow & \downarrow{\scriptstyle f^{\mathbb{S}}} \\ & & SA, \end{array} \qquad \begin{array}{ccc} SC & \xrightarrow{g^{\mathbb{S}}} & SB \\ & {\scriptstyle (f^{\mathbb{S}}\cdot g)^{\mathbb{S}}}\searrow & \downarrow{\scriptstyle f^{\mathbb{S}}} \\ & & SA \end{array}$$

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The algebras for $\mathbb{S}$

An $\mathbb{S}$-algebra $\mathbb{B} = (B, (-)^{\mathbb{B}})$ consists of:

- An object B in $\mathbf{C}$;
- for every arrow $h : X \to B$ in $\mathbf{C}$, an extension $h^{\mathbb{B}} : SX \to B$;
  Subject to the commutativity of the diagrams
  (with $h : X \to B$ and $y : Y \to SX$):

$$\begin{array}{ccc} X & \xrightarrow{\eta X} & SX \\ & {\scriptstyle h} \searrow & \downarrow {\scriptstyle h^{\mathbb{B}}} \\ & & B, \end{array} \qquad \begin{array}{ccc} SY & \xrightarrow{y^{\mathbb{S}}} & SX \\ & {\scriptstyle (h^{\mathbb{B}} y)^{\mathbb{B}}} \searrow & \downarrow {\scriptstyle h^{\mathbb{B}}} \\ & & B. \end{array}$$

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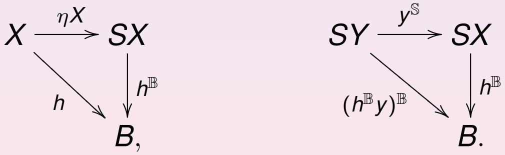

The algebras for $\mathbb{S}$

A morphism of $\mathbb{S}$-algebras $(B, (-)^{\mathbb{B}})$ to $(A, (-)^{\mathbb{A}})$ is

an arrow $\ell : B \to A$ in $\mathbf{C}$

subject to the commutativity of the diagram

$$\begin{array}{ccc} SX & \xrightarrow{h^{\mathbb{B}}} & B \\ & \searrow^{(\ell \cdot h)^{\mathbb{A}}} & \downarrow \ell \\ & & A. \end{array}$$

for every $h : X \to B$.

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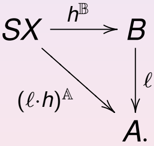

for every $h : X \to B$.

The algebras for $\mathbb{S}$

Theorem. The category of usual algebras for the monad $\mathbb{S}$ is isomorphic to the category of algebras just defined.

$\mathbb{S} = (S, \eta_S, (-)^{\mathbb{S}})$, $\mathbb{T} = (T, \eta_T, (-)^{\mathbb{T}})$ monads on **C**.

A distributive law of $\mathbb{S}$ over $\mathbb{T}$ can be given as follows:

- For every A in **C** an $\mathbb{S}$-algebra $(TSA, (-)^{\lambda})$

Subject to the axioms

- for every A in **C**, $(T\eta_S A \cdot \eta_T A)^{\lambda} = \eta_T SA$;
- for every $f : B \to TSA$,
  $(f^{\lambda})^{\mathbb{T}} : (TSB, (-)^{\lambda}) \to (TSA, (-)^{\lambda})$
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All the diagrams together

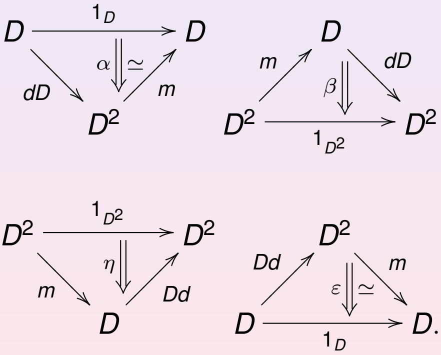
$f : B \to TSA,$
$g : C \to SB,$
$h : C \to TSB.$

$$\begin{array}{ccc} B & \xrightarrow{\eta_S B} & SB \\ & {\scriptstyle f}\searrow & \downarrow {\scriptstyle f^\lambda} \\ & & TSA; \end{array} \qquad \begin{array}{ccc} SC & \xrightarrow{g^{\mathbb{S}}} & SB \\ & {\scriptstyle (f^\lambda g)^\lambda}\searrow & \downarrow {\scriptstyle f^\lambda} \\ & & TSA; \end{array}$$

$$\begin{array}{ccc} A & \xrightarrow{\eta_S A} & SA \\ {\scriptstyle \eta_T A}\searrow & & \\ TA & & \downarrow {\scriptstyle \eta_T A = (T\eta_S A \cdot \eta_T A)^\lambda} \\ & {\scriptstyle T\eta_S A}\searrow & \\ & & TSA; \end{array} \qquad \begin{array}{ccc} SC & \xrightarrow{h^\lambda} & TSB \\ & {\scriptstyle ((f^\lambda)^{\mathbb{T}} h)^\lambda}\searrow & \downarrow {\scriptstyle (f^\lambda)^{\mathbb{T}}} \\ & & TSA. \end{array}$$

Colax idempotent monads

Colax idempotent monad
$\mathbb{D} = (D, d, m, \alpha, \beta, \eta, \varepsilon)$
on the 2-category $\mathcal{K}$, is given by $dD \dashv m \dashv Dd$:



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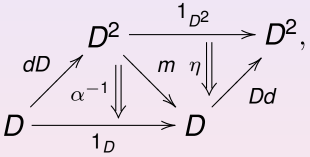
$\delta : dD \to Dd$



Colax idempotent monads

Theorem

A colax idempotent monad $\mathbb{D}$ on the 2-category $\mathcal{K}$ can be given as:

i. A *function* $D : \mathrm{Ob}(\mathcal{K}) \to \mathrm{Ob}(\mathcal{K})$.

ii. For every $\mathbf{A} \in \mathcal{K}$, a 1-cell $d\mathbf{A} : \mathbf{A} \to D\mathbf{A}$.

iii. For every 1-cell $F : \mathbf{B} \to D\mathbf{A}$, a right Kan extension of F along $d\mathbf{B}$

with $\mathbb{D}_F$ invertible.

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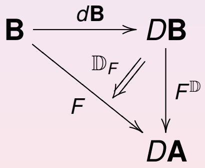
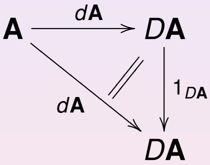


with $\mathbb{D}_F$ invertible.

Colax idempotent monads

Subject to the axioms

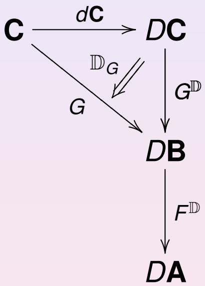
a. For every $\mathbf{A}$ in $\mathcal{K}$,



exhibits $1_{D\mathbf{A}}$ as a right Kan extension of $d\mathbf{A}$ along $d\mathbf{A}$.

Colax idempotent monads

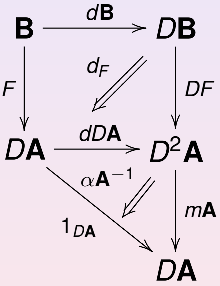
b. For every $G : \mathbf{C} \to D\mathbf{B}$ and $F : \mathbf{B} \to D\mathbf{A}$ the 2-cell



exhibits $F^{\mathbb{D}}G^{\mathbb{D}}$ as a right Kan extension of $F^{\mathbb{D}}G$ along $d\mathbf{C}$.

Proof

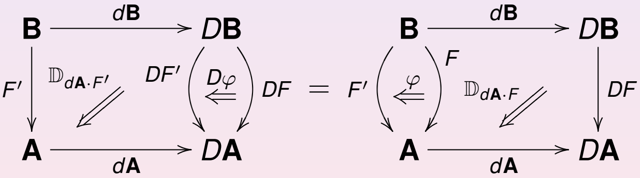
Assume $\mathbb{D} = (D, d, m, \alpha, \beta, \eta, \varepsilon)$ is a colax idempotent monad. Then



exhibits $F^{\mathbb{D}} = m\mathbf{A} \circ DF$ as a right Kan extension of F along $d\mathbf{B}$.

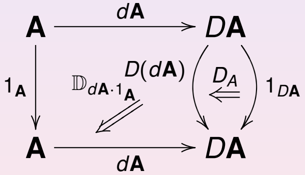
Colax idempotent monads

In the opposite direction.
First we must define a pseudofunctor $D : \mathcal{K} \to \mathcal{K}$.
For $F : \mathbf{B} \to \mathbf{A}$, $DF := (d\mathbf{A} \circ F)^{\mathbb{D}}$, $d_F := \mathbb{D}_{d\mathbf{A}\cdot F}$.
For $\varphi : F \to F' : \mathbf{B} \to \mathbf{A}$, $D\varphi$ is the unique 2-cell such that



Colax idempotent monads

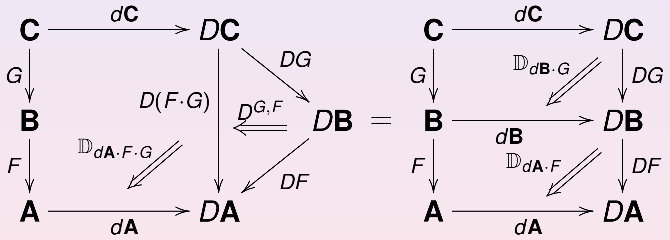
For $\mathbf{A}$ in $\mathcal{K}$,
define $D_A : 1_{D\mathbf{A}} \to D(1_{\mathbf{A}})$
as the unique 2-cell such that



$= 1_{d\mathbf{A}}$.

Colax idempotent monads

For $F : \mathbf{B} \to \mathbf{A}$ and $G : \mathbf{C} \to \mathbf{B}$,
define $D^{G,F} : DF \cdot DG \to D(F \cdot G)$
as the unique 2-cell such that



Colax idempotent monads

Define $m : D^2 \to D$ such that for every $\mathbf{A}$,

$$m\mathbf{A} = 1_{D\mathbf{A}}{}^{\mathbb{D}}.$$

For $F : \mathbf{B} \to \mathbf{A}$,
Define $m_F : DF \cdot m\mathbf{B} \to m\mathbf{A} \cdot D^2 f$
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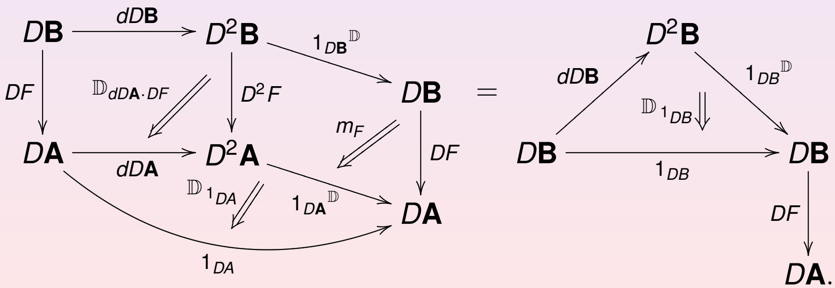
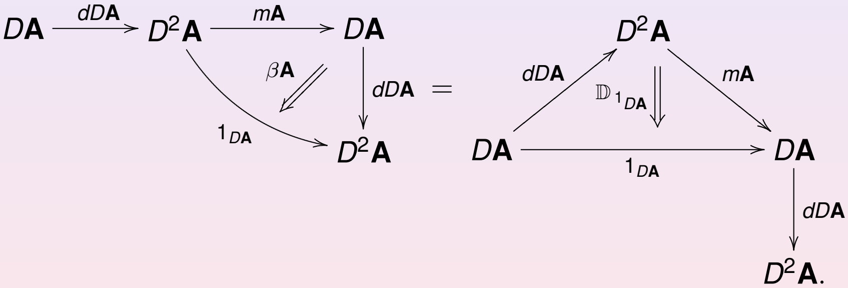


Colax idempotent monads

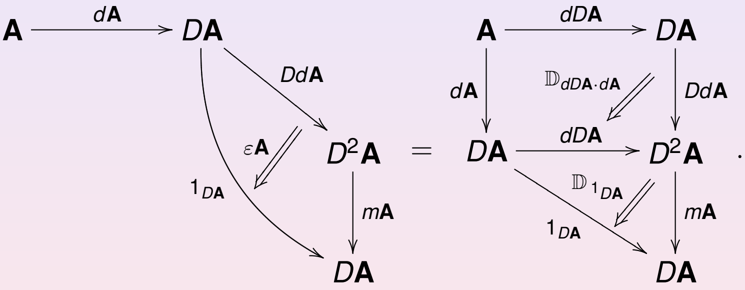
$\alpha\mathbf{A} = \mathbb{D}_{1_{D\mathbf{A}}}{}^{-1}$

$\beta\mathbf{A} : dD\mathbf{A} \cdot m\mathbf{A} \to 1_{D^2\mathbf{A}}$ as the unique 2-cell such that



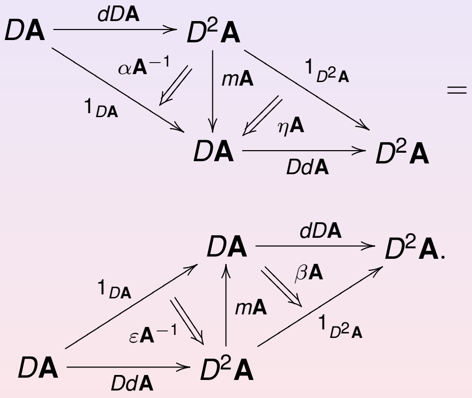
Colax idempotent monads

$\varepsilon : m\mathbf{A} \cdot Dd\mathbf{A} \to 1_{D\mathbf{A}}$ as the unique 2-cell such that



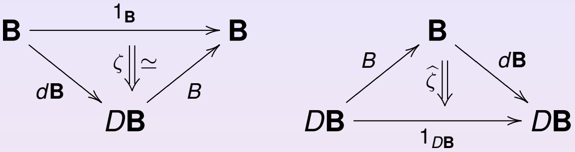
Colax idempotent monads

$\eta : 1_{D^2\mathbf{A}} \to Dd\mathbf{A} \cdot m\mathbf{A}$ as the unique 2-cell such that



Algebras for a colax idempotent monad

Recall that the algebras are adjunctions $\zeta, \widehat{\zeta} : B \dashv d\mathbf{B}$,



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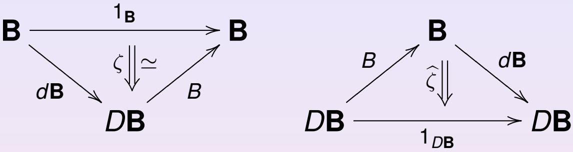

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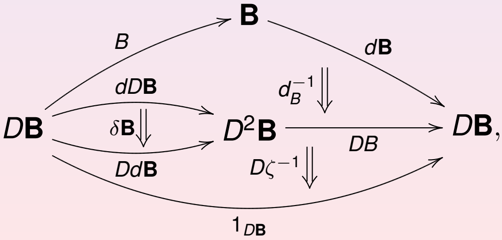
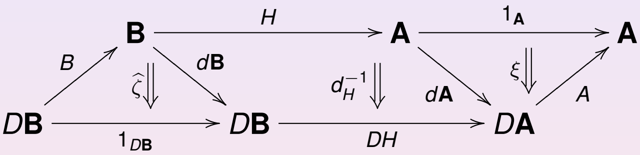


Algebras for a colax idempotent monad

A 1-cell from $\zeta : 1_{\mathbf{B}} \to B \cdot d\mathbf{B}$ to $\xi : 1_{\mathbf{A}} \to A \cdot d\mathbf{A}$ is a 1-cell $H : \mathbf{B} \to \mathbf{A}$ such that the the pasting



is invertible.

Given $H, K : \zeta \to \xi$, a 2-cell in $\mathbb{D}$-Alg is a 2-cell $\tau : H \to K$ in $\mathcal{K}$.

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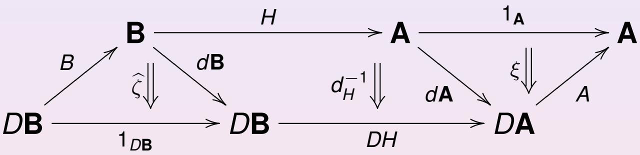

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An equivalent description of the algebras for $\mathbb{D}$

An object $\mathbb{B}$ consists of an object $\mathbf{B}$
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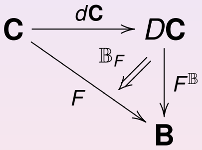

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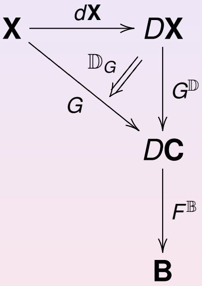

exhibits $F^{\mathbb{B}} \cdot G^{\mathbb{D}}$ as a right Kan extension of $F^{\mathbb{B}} \cdot G$ along $d\mathbf{X}$.

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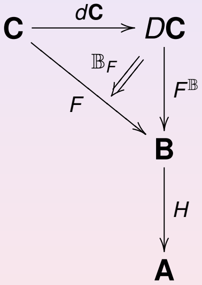

exhibits $F^{\mathbb{B}} \cdot H$ as a right Kan extension of $F \cdot H$ along $d\mathbf{C}$.
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Theorem. The categories of algebras just described are equivalent.

Distributive law for (co-)lax idempotent monads

$\mathbb{D}$ lax idempotent monad.
$\mathbb{U}$ colax idempotent monad.

Theorem. A distributive law of $\mathbb{U}$ over $\mathbb{D}$ can equivalently be given as follows.

- For every $\mathbf{A}$ in $\mathcal{K}$, a $\mathbb{U}$-algebra structure $(DU\mathbf{A}, (\ )^{\lambda})$.
- For every $\mathbf{A}$, $d_{u\mathbf{A}}^{-1}$ exhibits $dU\mathbf{A}$ as a right Kan extension of $Du\mathbf{A} \circ d\mathbf{A}$ along $u\mathbf{A}$.
- For every $H : \mathbf{C} \to DU\mathbf{A}$, $(H^{\lambda})^{\mathbb{D}} : (DU\mathbf{C}, (\ )^{\lambda}) \to (DU\mathbf{A}, (\ )^{\lambda})$ is an algebra morphism.

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Definition. A *Manes' presentation* of a pseudomonad $\mathbb{D}$ on $\mathcal{A}$ consists of

- A function $D : \mathrm{Ob}(\mathcal{A}) \to \mathrm{Ob}(\mathcal{A})$.
- For every $A \in \mathcal{A}$, a 1-cell $dA : A \to DA$.
- For every $A, B \in \mathcal{A}$, a *functor* $(\ )^{\mathbb{D}} : \mathcal{A}(A, DB) \to \mathcal{A}(DA, DB)$.
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Pseudomonads

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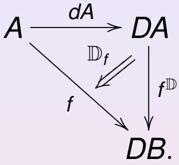

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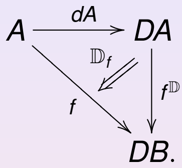

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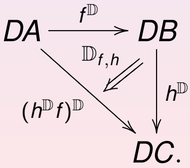
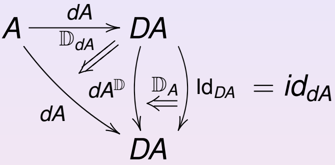


Subject to the coherence conditions

- For every $A \in \mathcal{A}$,



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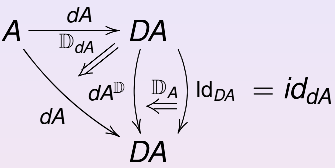

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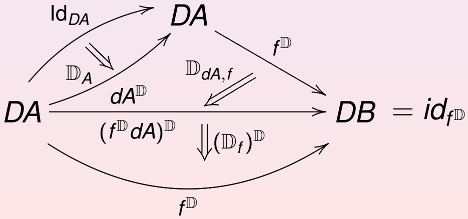

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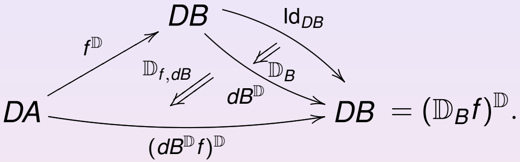

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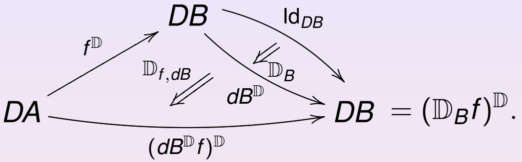

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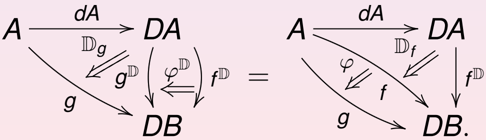

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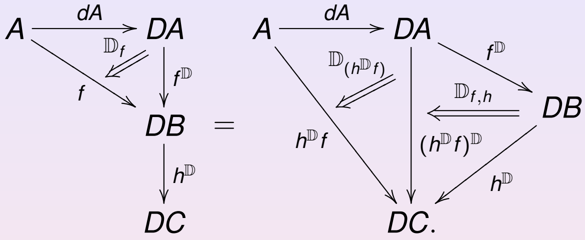

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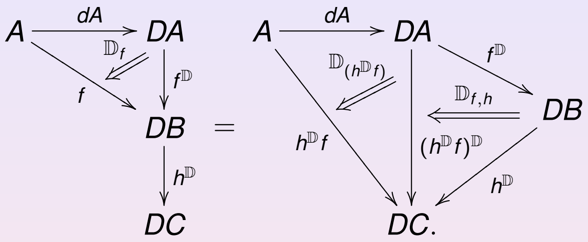

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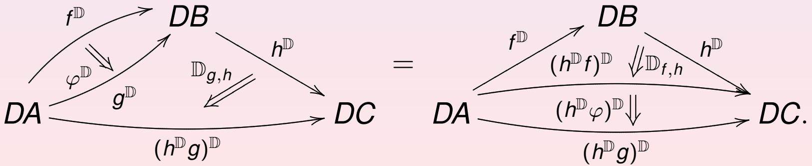

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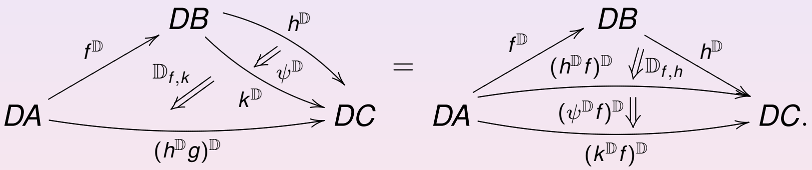
- For every $f : A \to DB$ and every $\psi : h \to k : B \to DC$,



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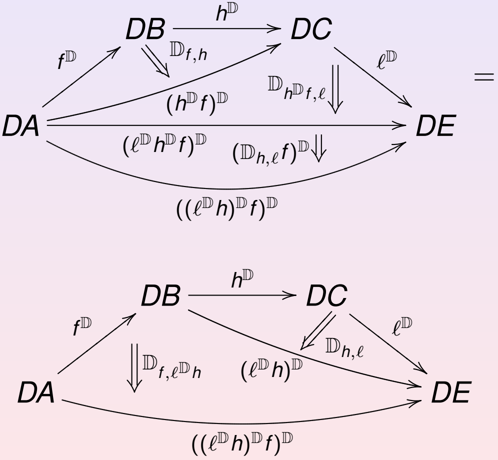


Proposition. We induce a pseudofunctor $D\colon \mathcal{A} \to \mathcal{A}$:
D as given on objects.
For objects A, B we define $D_{A,B}$ as the composite

$$\mathcal{A}(A,B) \xrightarrow{-\circ dB} \mathcal{A}(A,DB) \xrightarrow{(\ )^{\mathbb{D}}} \mathcal{A}(DA,DB).$$

For every A, $D_A = \mathbb{D}_A$.
For $f\colon A \to B$ and $h\colon B \to C$, define
$D^{f,h}\colon Dh\,Df \to D(hf)$ as $(\mathbb{D}_{dC\,h}\,f)^{\mathbb{D}} \cdot \mathbb{D}_{dB\,f,dC\,h}$.

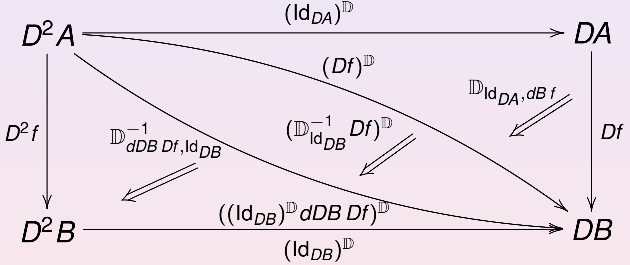
Pseudomonads

The rest of the structure for a usual pseudomonad is:
For every $f : A \to B$ we define $d_f := \mathbb{D}_{dB\,f}$:

$$\begin{array}{ccc} A & \xrightarrow{dA} & DA \\ {\scriptstyle f}\downarrow & \overset{d_f}{\swArrow} & \downarrow{\scriptstyle (dB\,f)^{\mathbb{D}} = Df} \\ B & \xrightarrow[dB]{} & DB. \end{array}$$

Pseudomonads

For every A we define $mA = (\mathrm{Id}_{DA})^{\mathbb{D}}$ and for every $f : A \to B$ we define m_f as



Pseudomonads

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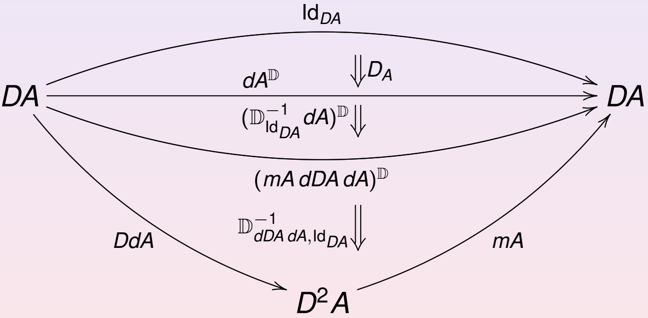
ηA as



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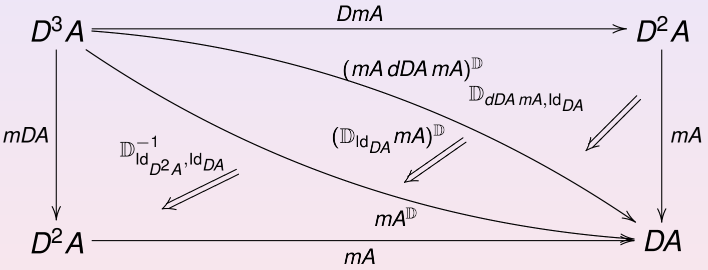

An equivalent definition of the algebras

An algebra for the pseudomonad $\mathbb{D}$ consists of

- An object A in $\mathcal{A}$.
- For every X in $\mathcal{A}$ a functor $(\ )^{\mathbb{A}} : \mathcal{A}(X, A) \to \mathcal{A}(DX, A)$.
- For every $h : X \to A$ an invertible 2-cell

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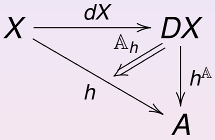

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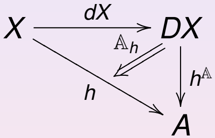

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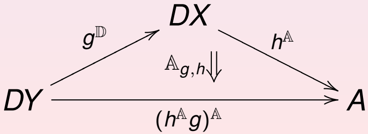

An equivalent definition of the algebras

Subject to the axioms

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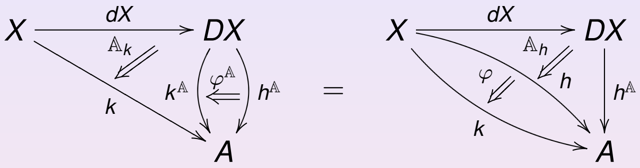

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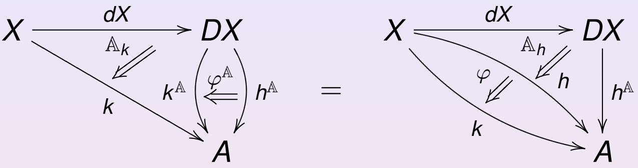

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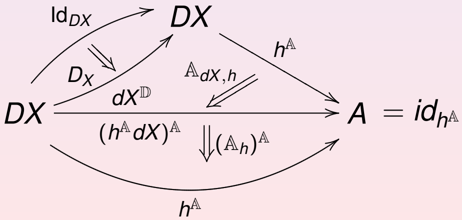

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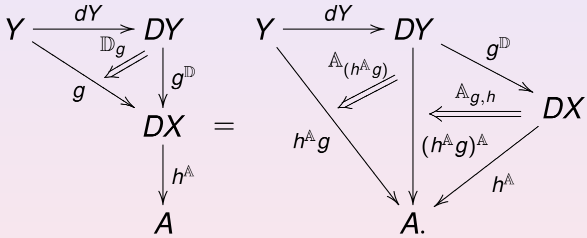

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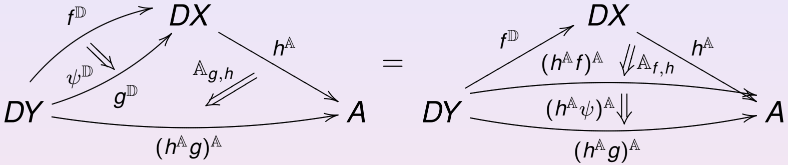

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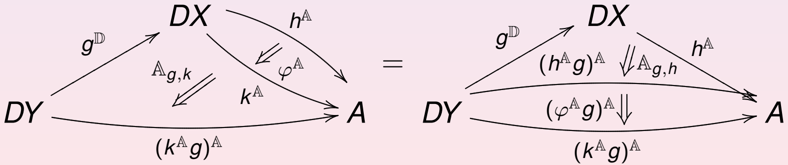

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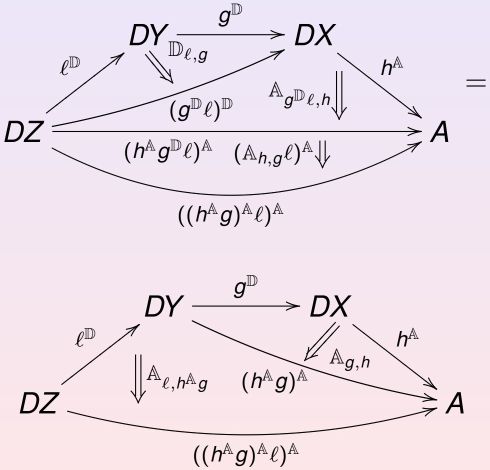

An equivalent definition of the algebras

Given $\mathbb{A} = (A, (\ )^{\mathbb{A}})$ and $\mathbb{B} = (B, (\ )^{\mathbb{B}})$
a 1-cell $\mathbb{A} \longrightarrow \mathbb{B}$
is a 1-cell $f : A \longrightarrow B$ in $\mathcal{A}$
together with an invertible 2-cell

for every $h : X \longrightarrow A$ in $\mathcal{A}$.

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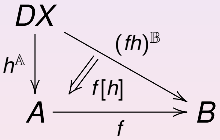

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An equivalent definition of the algebras

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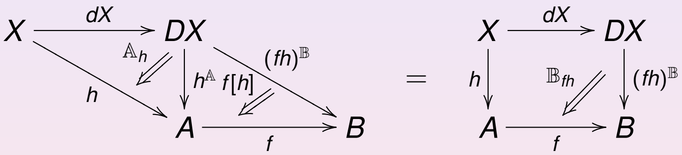
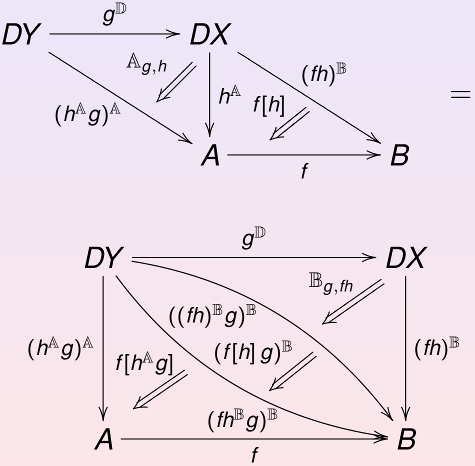
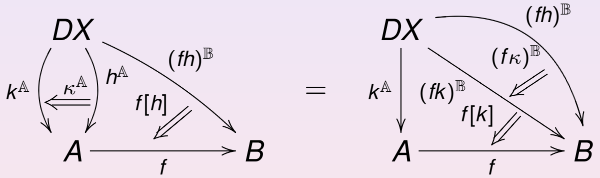


for any $g : Y \to DX$

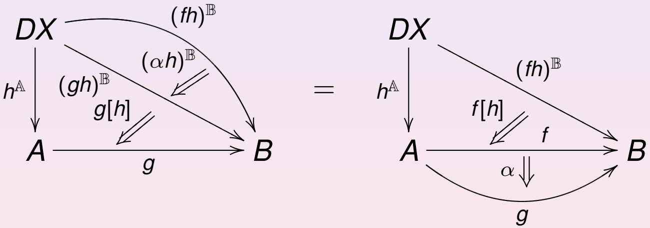


An equivalent definition of the algebras

and for every $\kappa : h \to k : X \to A$,



A 2-cell $\alpha : f \to g : \mathbb{A} \to \mathbb{B}$
is a 2-cell $\alpha : f \to g$
such that for every $h : X \to A$ we have



An equivalent definition of the algebras

Teorema. The usual bicategory of algebras for $\mathbb{D}$ is equivalent to the bicategory of algebras just defined.