

# Pseudomonads — No iteration version

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# Manes' exercise on monads

**Exercise 1.3.12** A monad  $\mathbb{S}$  on a category  $\mathbf{C}$  is equivalent to:

- A function  $|S| : |\mathbf{C}| \rightarrow |\mathbf{C}|$ ;
- for every  $A \in \mathbf{C}$ , an arrow  $\eta A : A \rightarrow SA$ ;
- for every morphism  $f : B \rightarrow SA$  in  $\mathbf{C}$ , an  $\mathbb{S}$ -extension  $f^{\mathbb{S}} : SB \rightarrow SA$ .

Subject to the axioms:

- for every  $A$  in  $\mathbf{C}$ ,

$$(\eta A)^{\mathbb{S}} = 1_{SA};$$

- for every  $f : B \rightarrow SA$  in  $\mathbf{C}$  and  $g : C \rightarrow SB$ , the diagrams

$$\begin{array}{ccc} B & \xrightarrow{\eta B} & SB \\ & \searrow f & \downarrow f^{\mathbb{S}} \\ & & SA \end{array} \qquad \begin{array}{ccc} SC & \xrightarrow{g^{\mathbb{S}}} & SB \\ & \searrow (f^{\mathbb{S}} \cdot g)^{\mathbb{S}} & \downarrow f^{\mathbb{S}} \\ & & SA \end{array}$$

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# The algebras for $\mathbb{S}$

An  $\mathbb{S}$ -algebra  $\mathbb{B} = (B, (-)^{\mathbb{B}})$  consists of:

- An object  $B$  in  $\mathbf{C}$ ;
- for every arrow  $h: X \rightarrow B$  in  $\mathbf{C}$ , an extension  $h^{\mathbb{B}}: SX \rightarrow B$ ;  
Subject to the commutativity of the diagrams  
(with  $h: X \rightarrow B$  and  $y: Y \rightarrow SX$ ):

$$\begin{array}{ccc} X & \xrightarrow{\eta^X} & SX \\ & \searrow h & \downarrow h^{\mathbb{B}} \\ & & B, \end{array}$$

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A morphism of  $\mathbb{S}$ -algebras  $(B, (-)^{\mathbb{B}})$  to  $(A, (-)^{\mathbb{A}})$  is

an arrow  $\ell : B \rightarrow A$  in  $\mathbf{C}$

subject to the commutativity of the diagram

$$\begin{array}{ccc} SX & \xrightarrow{h^{\mathbb{B}}} & B \\ & \searrow^{(\ell \cdot h)^{\mathbb{A}}} & \downarrow \ell \\ & & A. \end{array}$$

for every  $h : X \rightarrow B$ .

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**Theorem.** The category of usual algebras for the monad  $\mathbb{S}$  is isomorphic to the category of algebras just defined.



$\mathbb{S} = (\mathbf{S}, \eta_{\mathbf{S}}, (-)^{\mathbb{S}})$ ,  $\mathbb{T} = (\mathbf{T}, \eta_{\mathbf{T}}, (-)^{\mathbb{T}})$  monads on  $\mathbf{C}$ .

A distributive law of  $\mathbb{S}$  over  $\mathbb{T}$  can be given as follows:

- For every  $A$  in  $\mathbf{C}$  an  $\mathbb{S}$ -algebra  $(TSA, (-)^{\lambda})$

Subject to the axioms

- for every  $A$  in  $\mathbf{C}$ ,  $(T\eta_{\mathbf{S}}A \cdot \eta_{\mathbf{T}}A)^{\lambda} = \eta_{\mathbf{T}}SA$ ;
- for every  $f : B \rightarrow TSA$ ,  
 $(f^{\lambda})^{\mathbb{T}} : (TSB, (-)^{\lambda}) \rightarrow (TSA, (-)^{\lambda})$   
is a morphism of  $\mathbb{S}$ -algebras.

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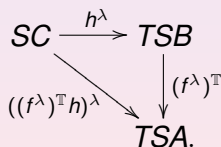
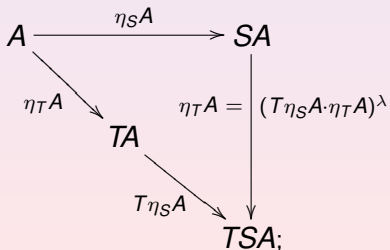
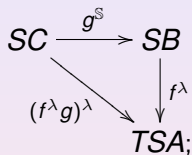
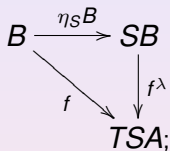
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# All the diagrams together

$f: B \rightarrow TSA,$   
 $g: C \rightarrow SB,$   
 $h: C \rightarrow TSB.$



# Colax idempotent monads

Colax idempotent monad

$$\mathbb{D} = (D, d, m, \alpha, \beta, \eta, \varepsilon)$$

on the 2-category  $\mathcal{K}$ , is given by  $dD \dashv m \dashv Dd$ :

$$\begin{array}{ccc} D & \xrightarrow{1_D} & D \\ & \searrow dD & \swarrow m \\ & D^2 & \end{array} \quad \alpha \Downarrow \simeq$$

$$\begin{array}{ccc} & D & \\ m \nearrow & & \searrow dD \\ D^2 & \xrightarrow{1_{D^2}} & D^2 \end{array} \quad \beta \Downarrow$$

$$\begin{array}{ccc} D^2 & \xrightarrow{1_{D^2}} & D^2 \\ & \searrow m & \swarrow Dd \\ & D & \end{array} \quad \eta \Downarrow$$

$$\begin{array}{ccc} & D^2 & \\ Dd \nearrow & & \searrow m \\ D & \xrightarrow{1_D} & D \end{array} \quad \varepsilon \Downarrow \simeq$$

# Colax idempotent monads

$$\delta : dD \rightarrow Dd$$

A commutative diagram illustrating the naturality of the  $\delta$  natural transformation. The diagram consists of the following nodes and arrows:

- Top-left node:  $D^2$
- Top-right node:  $D^2,$
- Bottom-left node:  $D$
- Bottom-right node:  $D$

The arrows are:

- $D \xrightarrow{dD} D^2$  (diagonal arrow from bottom-left to top-left)
- $D \xrightarrow{1_D} D$  (horizontal arrow from bottom-left to bottom-right)
- $D^2 \xrightarrow{1_{D^2}} D^2,$  (horizontal arrow from top-left to top-right)
- $D^2 \xrightarrow{m} D$  (diagonal arrow from top-left to bottom-right)
- $D \xrightarrow{Dd} D^2,$  (diagonal arrow from bottom-right to top-right)
- $\alpha^{-1} : D^2 \Downarrow D$  (vertical arrow from top-left to bottom-left)
- $\eta : D \Downarrow D^2,$  (vertical arrow from bottom-right to top-right)

The diagram is commutative, meaning the composition of arrows along any path between two nodes is the same. For example, the path  $D \xrightarrow{dD} D^2 \xrightarrow{m} D$  is equal to the path  $D \xrightarrow{1_D} D$ . Similarly, the path  $D \xrightarrow{Dd} D^2 \xrightarrow{1_{D^2}} D^2,$  is equal to the path  $D \xrightarrow{Dd} D^2,$ .

# Colax idempotent monads

## Theorem

A colax idempotent monad  $\mathbb{D}$  on the 2-category  $\mathcal{K}$  can be given as:

- A function  $D: \text{Ob}(\mathcal{K}) \rightarrow \text{Ob}(\mathcal{K})$ .
- For every  $\mathbf{A} \in \mathcal{K}$ , a 1-cell  $d\mathbf{A}: \mathbf{A} \rightarrow D\mathbf{A}$ .
- For every 1-cell  $F: \mathbf{B} \rightarrow D\mathbf{A}$ , a right Kan extension of  $F$  along  $d\mathbf{B}$

$$\begin{array}{ccc} \mathbf{B} & \xrightarrow{d\mathbf{B}} & D\mathbf{B} \\ & \searrow F & \downarrow F^{\mathbb{D}} \\ & & D\mathbf{A} \end{array}$$

$\Downarrow \mathbb{D}_F$

with  $\mathbb{D}_F$  invertible.



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- iii. For every 1-cell  $F: \mathbf{B} \rightarrow D\mathbf{A}$ , a right Kan extension of  $F$  along  $d\mathbf{B}$

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$$\begin{array}{ccc} \mathbf{B} & \xrightarrow{d\mathbf{B}} & D\mathbf{B} \\ & \searrow F & \downarrow F^{\mathbb{D}} \\ & & D\mathbf{A} \end{array}$$

The diagram shows a commutative triangle. The top horizontal arrow is labeled  $d\mathbf{B}$ . The right vertical arrow is labeled  $F^{\mathbb{D}}$ . The diagonal arrow from  $\mathbf{B}$  to  $D\mathbf{A}$  is labeled  $F$ . A double-lined arrow labeled  $\mathbb{D}_F$  points from the top horizontal arrow  $d\mathbf{B}$  to the diagonal arrow  $F$ .

with  $\mathbb{D}_F$  invertible.

# Colax idempotent monads

Subject to the axioms

- a. For every  $\mathbf{A}$  in  $\mathcal{K}$ ,

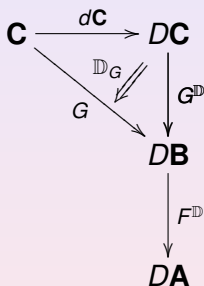
$$\begin{array}{ccc} \mathbf{A} & \xrightarrow{d\mathbf{A}} & D\mathbf{A} \\ & \searrow^{d\mathbf{A}} & \downarrow 1_{D\mathbf{A}} \\ & & D\mathbf{A} \end{array}$$

The diagram shows a commutative triangle. The top horizontal arrow is labeled  $d\mathbf{A}$ . The bottom horizontal arrow is also labeled  $d\mathbf{A}$ . The right vertical arrow is labeled  $1_{D\mathbf{A}}$ . A double arrow  $\parallel$  indicates that the two paths from  $\mathbf{A}$  to  $D\mathbf{A}$  are equal.

exhibits  $1_{D\mathbf{A}}$  as a right Kan extension of  $d\mathbf{A}$  along  $d\mathbf{A}$ .

# Colax idempotent monads

b. For every  $G : \mathbf{C} \rightarrow \mathbf{DB}$  and  $F : \mathbf{B} \rightarrow \mathbf{DA}$  the 2-cell



exhibits  $F^{\mathbb{D}} G^{\mathbb{D}}$  as a right Kan extension of  $F^{\mathbb{D}} G$  along  $d\mathbf{C}$ .

# Colax idempotent monads

*Proof*

Assume  $\mathbb{D} = (D, d, m, \alpha, \beta, \eta, \varepsilon)$  is a colax idempotent monad.

Then

$$\begin{array}{ccc} \mathbf{B} & \xrightarrow{d\mathbf{B}} & D\mathbf{B} \\ \downarrow F & \swarrow d_F & \downarrow DF \\ D\mathbf{A} & \xrightarrow{dD\mathbf{A}} & D^2\mathbf{A} \\ & \searrow \alpha\mathbf{A}^{-1} & \downarrow m\mathbf{A} \\ & & D\mathbf{A} \\ & \swarrow 1_{D\mathbf{A}} & \\ & & \end{array}$$

exhibits  $F^{\mathbb{D}} = m\mathbf{A} \circ DF$  as a right Kan extension of  $F$  along  $d\mathbf{B}$ .

# Colax idempotent monads

In the opposite direction.

First we must define a pseudofunctor  $D: \mathcal{K} \rightarrow \mathcal{K}$ .

For  $F: \mathbf{B} \rightarrow \mathbf{A}$ ,  $DF := (d\mathbf{A} \circ F)^{\mathbb{D}}$ ,  $d_F := \mathbb{D}_{d\mathbf{A} \cdot F}$ .

For  $\varphi: F \rightarrow F': \mathbf{B} \rightarrow \mathbf{A}$ ,  $D\varphi$  is the unique 2-cell such that

$$\begin{array}{ccc}
 \mathbf{B} & \xrightarrow{d\mathbf{B}} & D\mathbf{B} \\
 \downarrow F' & \mathbb{D}_{d\mathbf{A} \cdot F'} & \downarrow DF' \\
 \mathbf{A} & \xrightarrow{d\mathbf{A}} & D\mathbf{A}
 \end{array}
 \quad
 \begin{array}{c}
 \left( \begin{array}{c} \mathbf{B} \\ \leftarrow \varphi \\ \mathbf{A} \end{array} \right)^F \\
 \leftarrow D\varphi \\
 \left( \begin{array}{c} D\mathbf{B} \\ \leftarrow DF \\ D\mathbf{A} \end{array} \right)
 \end{array}
 \quad
 =
 \quad
 \begin{array}{ccc}
 \mathbf{B} & \xrightarrow{d\mathbf{B}} & D\mathbf{B} \\
 \downarrow F & \mathbb{D}_{d\mathbf{A} \cdot F} & \downarrow DF \\
 \mathbf{A} & \xrightarrow{d\mathbf{A}} & D\mathbf{A}
 \end{array}$$

# Colax idempotent monads

For  $\mathbf{A}$  in  $\mathcal{K}$ ,  
define  $D_A: 1_{DA} \rightarrow D(1_A)$   
as the unique 2-cell such that

$$\begin{array}{ccc} \mathbf{A} & \xrightarrow{d_A} & D\mathbf{A} \\ \downarrow 1_A & \Downarrow \mathbb{D}_{d_A \cdot 1_A} & \left( \begin{array}{c} \text{curved arrow} \\ \leftarrow D_A \end{array} \right) \\ \mathbf{A} & \xrightarrow{d_A} & D\mathbf{A} \end{array} \quad 1_{DA} = 1_{d_A}.$$



# Colax idempotent monads

For  $F: \mathbf{B} \rightarrow \mathbf{A}$  and  $G: \mathbf{C} \rightarrow \mathbf{B}$ ,  
 define  $D^{G,F}: DF \cdot DG \rightarrow D(F \cdot G)$   
 as the unique 2-cell such that

$$\begin{array}{ccc}
 \mathbf{C} & \xrightarrow{d\mathbf{C}} & \mathbf{DC} \\
 G \downarrow & & \searrow DG \\
 \mathbf{B} & & \mathbf{DB} \\
 F \downarrow & \mathbb{D}_{d\mathbf{A} \cdot F \cdot G} \swarrow & \leftarrow D^{G,F} \\
 \mathbf{A} & \xrightarrow{d\mathbf{A}} & \mathbf{DA} \\
 & & \swarrow DF
 \end{array}
 \quad
 \begin{array}{ccc}
 \mathbf{C} & \xrightarrow{d\mathbf{C}} & \mathbf{DC} \\
 G \downarrow & \mathbb{D}_{d\mathbf{B} \cdot G} \swarrow & \searrow DG \\
 \mathbf{B} & \xrightarrow{d\mathbf{B}} & \mathbf{DB} \\
 F \downarrow & \mathbb{D}_{d\mathbf{A} \cdot F} \swarrow & \searrow DF \\
 \mathbf{A} & \xrightarrow{d\mathbf{A}} & \mathbf{DA}
 \end{array}
 =$$

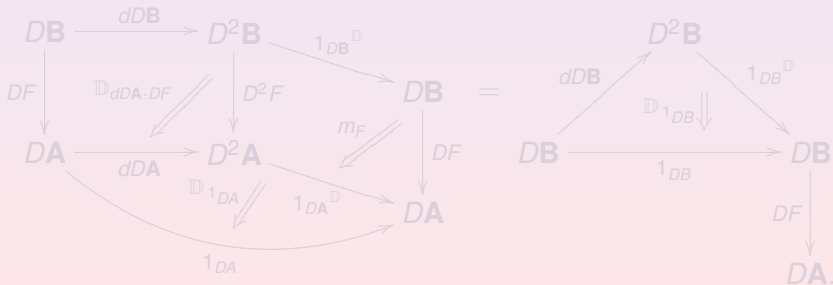
# Colax idempotent monads

Define  $m : D^2 \rightarrow D$  such that for every  $\mathbf{A}$ ,

$$m\mathbf{A} = 1_{D\mathbf{A}}^{\mathbb{D}}.$$

For  $F : \mathbf{B} \rightarrow \mathbf{A}$ ,

Define  $m_F : DF \cdot m\mathbf{B} \rightarrow m\mathbf{A} \cdot D^2f$   
as the unique 2-cell such that



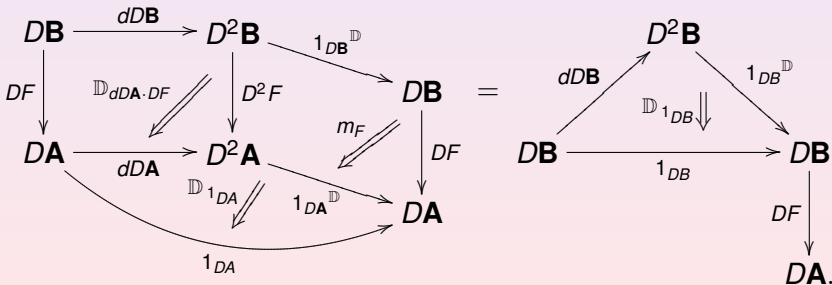
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# Colax idempotent monads

$$\alpha_{\mathbf{A}} = \mathbb{D}_{1_{DA}}^{-1}$$

$\beta_{\mathbf{A}} : dDA \cdot m_{\mathbf{A}} \rightarrow 1_{D^2\mathbf{A}}$  as the unique 2-cell such that

$$\begin{array}{ccc}
 DA & \xrightarrow{dDA} & D^2\mathbf{A} \xrightarrow{m_{\mathbf{A}}} DA \\
 & & \searrow \beta_{\mathbf{A}} \swarrow \\
 & & D^2\mathbf{A} \xrightarrow{dDA} DA \\
 & \searrow 1_{DA} & \\
 & & D^2\mathbf{A}
 \end{array}
 =
 \begin{array}{ccc}
 & & D^2\mathbf{A} \\
 & \nearrow dDA & \searrow m_{\mathbf{A}} \\
 DA & \xrightarrow{1_{DA}} & DA \\
 & & \downarrow dDA \\
 & & D^2\mathbf{A}
 \end{array}$$

# Colax idempotent monads

$\varepsilon : m\mathbf{A} \cdot Dd\mathbf{A} \rightarrow 1_{D\mathbf{A}}$  as the unique 2-cell such that

$$\begin{array}{ccc}
 \mathbf{A} & \xrightarrow{d\mathbf{A}} & D\mathbf{A} \\
 & & \searrow Dd\mathbf{A} \\
 & & D^2\mathbf{A} \\
 & \searrow \varepsilon\mathbf{A} & \swarrow \\
 & & D\mathbf{A} \\
 & \searrow 1_{D\mathbf{A}} & \swarrow \\
 & & D\mathbf{A} \\
 & & \downarrow m\mathbf{A} \\
 & & D\mathbf{A}
 \end{array}
 =
 \begin{array}{ccc}
 \mathbf{A} & \xrightarrow{dD\mathbf{A}} & D\mathbf{A} \\
 \downarrow d\mathbf{A} & & \downarrow Dd\mathbf{A} \\
 D\mathbf{A} & \xrightarrow{dD\mathbf{A}} & D^2\mathbf{A} \\
 \searrow 1_{D\mathbf{A}} & \swarrow \mathbb{D}_{1_{D\mathbf{A}}} & \downarrow m\mathbf{A} \\
 & & D\mathbf{A}
 \end{array}
 .$$

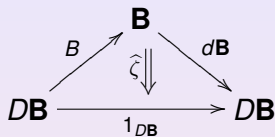
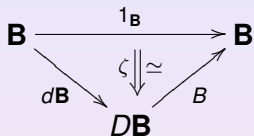
# Colax idempotent monads

$\eta : 1_{D^2\mathbf{A}} \rightarrow Dd\mathbf{A} \cdot m\mathbf{A}$  as the unique 2-cell such that

$$\begin{array}{ccc}
 DA & \xrightarrow{dDA} & D^2A \\
 & \searrow 1_{DA} & \downarrow mA \\
 & & DA \\
 & & \swarrow DdA \\
 & & D^2A
 \end{array}
 \quad = \quad
 \begin{array}{ccc}
 & & DA \\
 & \nearrow 1_{DA} & \downarrow \beta A \\
 & & D^2A \\
 DA & \xrightarrow{DdA} & D^2A
 \end{array}$$

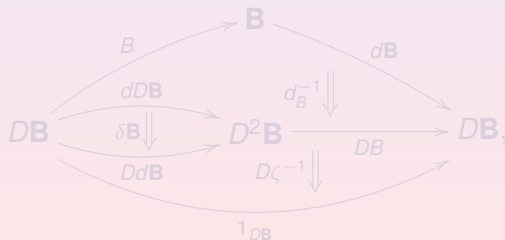
# Algebras for a colax idempotent monad

Recall that the algebras are adjunctions  $\zeta, \hat{\zeta} : B \dashv dB$ ,



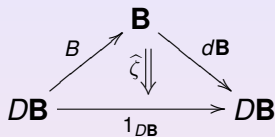
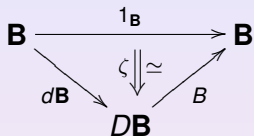
with invertible unit.

$\hat{\zeta}$  is completely determined by  $\zeta$  as the pasting



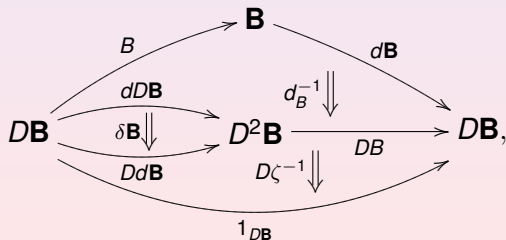
# Algebras for a colax idempotent monad

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# Algebras for a colax idempotent monad

A 1-cell from  $\zeta : 1_{\mathbf{B}} \rightarrow B \cdot d\mathbf{B}$  to  $\xi : 1_{\mathbf{A}} \rightarrow A \cdot d\mathbf{A}$  is a 1-cell  $H : \mathbf{B} \rightarrow \mathbf{A}$  such that the the pasting

$$\begin{array}{ccccc}
 & & \mathbf{B} & \xrightarrow{H} & \mathbf{A} & \xrightarrow{1_{\mathbf{A}}} & \mathbf{A} \\
 & \nearrow B & \downarrow \widehat{\zeta} & \searrow dB & \downarrow d_H^{-1} & \searrow dA & \downarrow \xi \\
 DB & \xrightarrow{1_{DB}} & DB & \xrightarrow{DH} & DA & \nearrow A & \\
 & & & & & & 
 \end{array}$$

is invertible.

Given  $H, K : \zeta \rightarrow \xi$ , a 2-cell in  $\mathbb{D}\text{-Alg}$   
 is a 2-cell  $\tau : H \rightarrow K$  in  $\mathcal{K}$ .

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 DB & \xrightarrow{1_{DB}} & DB & \xrightarrow{DH} & DA & \nearrow A & \\
 & & & & & & 
 \end{array}$$

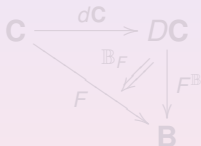
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# An equivalent description of the algebras for $\mathbb{D}$

An object  $\mathbb{B}$  consists of an object  $\mathbf{B}$

together with an assignment, to every  $F : \mathbf{C} \rightarrow \mathbf{B}$ ,  
of a right Kan extension  $F^{\mathbb{B}} : D\mathbf{C} \rightarrow \mathbf{B}$  of  $F$  along  $d\mathbf{C}$



with  $\mathbb{B}_F$  invertible,

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$\mathbb{B}_F$  (represented by two parallel arrows from  $D\mathbf{C}$  to  $\mathbf{B}$ )

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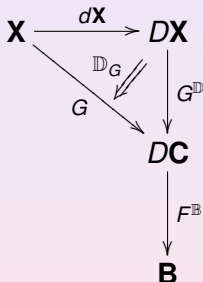
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such that for every  $G: \mathbf{X} \rightarrow DC$ ,  
the diagram



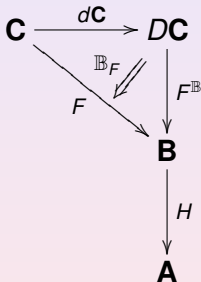
exhibits  $F^{\mathbb{B}} \cdot G^{\mathbb{D}}$  as a right Kan extension of  $F^{\mathbb{B}} \cdot G$  along  $d\mathbf{X}$ .

# An equivalent description of the algebras for $\mathbb{D}$

A 1-cell  $H: \mathbb{B} \rightarrow \mathbb{A}$

is a 1-cell  $H: \mathbf{B} \rightarrow \mathbf{A}$  in  $\mathcal{K}$

such that for every  $F: \mathbf{C} \rightarrow \mathbf{B}$ , the diagram



exhibits  $F^{\mathbb{B}} \cdot H$  as a right Kan extension of  $F \cdot H$  along  $d\mathbf{C}$ .

A 2-cell  $\tau: H \rightarrow K: \mathbb{B} \rightarrow \mathbb{A}$  is a 2-cell  $\tau: H \rightarrow K$  in  $\mathcal{K}$ .

# An equivalent description of the algebras for $\mathbb{D}$

**Theorem.** The categories of algebras just described are equivalent.



# Distributive law for (co-)lax idempotent monads

$\mathbb{D}$  lax idempotent monad.

$\mathbb{U}$  colax idempotent monad.

**Theorem.** A distributive law of  $\mathbb{U}$  over  $\mathbb{D}$  can equivalently be given as follows.

- For every  $\mathbf{A}$  in  $\mathcal{K}$ , a  $\mathbb{U}$ -algebra structure  $(D\mathbf{U}\mathbf{A}, ( )^\lambda)$ .
- For every  $\mathbf{A}$ ,  $d_{u\mathbf{A}}^{-1}$  exhibits  $d\mathbf{U}\mathbf{A}$  as a right Kan extension of  $Du\mathbf{A} \circ d\mathbf{A}$  along  $u\mathbf{A}$ .
- For every  $H: \mathbf{C} \rightarrow D\mathbf{U}\mathbf{A}$ ,  $(H^\lambda)^\mathbb{D}: (D\mathbf{U}\mathbf{C}, ( )^\lambda) \rightarrow (D\mathbf{U}\mathbf{A}, ( )^\lambda)$  is an algebra morphism.

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- For every  $H: \mathbf{C} \rightarrow D\mathbf{U}\mathbf{A}$ ,  $(H^\lambda)^\mathbb{D}: (D\mathbf{U}\mathbf{C}, ( )^\lambda) \rightarrow (D\mathbf{U}\mathbf{A}, ( )^\lambda)$  is an algebra morphism.

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- For every  $H: \mathbf{C} \rightarrow DUA$ ,  $(H^\lambda)^\mathbb{D}: (DUC, ( )^\lambda) \rightarrow (DUA, ( )^\lambda)$  is an algebra morphism.

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**Definition.** A *Manes' presentation* of a pseudomonad  $\mathbb{D}$  on  $\mathcal{A}$  consists of

- A function  $D: \text{Ob}(\mathcal{A}) \rightarrow \text{Ob}(\mathcal{A})$ .
- For every  $A \in \mathcal{A}$ , a 1-cell  $dA: A \rightarrow DA$ .
- For every  $A, B \in \mathcal{A}$ , a functor  $(\ )^{\mathbb{D}}: \mathcal{A}(A, DB) \rightarrow \mathcal{A}(DA, DB)$ .
- For every  $A \in \mathcal{A}$ , an invertible 2-cell  $\mathbb{D}_A: \text{Id}_{DA} \rightarrow dA^{\mathbb{D}}$ .

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# Pseudomonads

- For every  $f : A \rightarrow DB$ , an invertible 2-cell

$$\begin{array}{ccc} A & \xrightarrow{dA} & DA \\ & \searrow f & \downarrow f^{\mathbb{D}} \\ & & DB \end{array}$$

$\mathbb{D}_f$  (invertible 2-cell)

- For every  $f : A \rightarrow DB$  and  $h : B \rightarrow DC$ , an invertible 2-cell

$$\begin{array}{ccc} DA & \xrightarrow{f^{\mathbb{D}}} & DB \\ & \searrow (h^{\mathbb{D}}f)^{\mathbb{D}} & \downarrow h^{\mathbb{D}} \\ & & DC \end{array}$$

$\mathbb{D}_{f,h}$  (invertible 2-cell)

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$\mathbb{D}_f$

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$$\begin{array}{ccc} DA & \xrightarrow{f^{\mathbb{D}}} & DB \\ & \searrow (h^{\mathbb{D}}f)^{\mathbb{D}} & \downarrow h^{\mathbb{D}} \\ & & DC \end{array}$$

$\mathbb{D}_{f,h}$

# Pseudomonads

Subject to the coherence conditions

- For every  $A \in \mathcal{A}$ ,

$$\begin{array}{ccc}
 A & \xrightarrow{dA} & DA \\
 & \Downarrow \mathbb{D}_{dA} & \\
 & dA^{\mathbb{D}} & \\
 & \Downarrow \mathbb{D}_A & \\
 & dA & \\
 & \Downarrow & \\
 & DA &
 \end{array}
 \quad \text{Id}_{DA} = id_{dA}$$

- For every  $f : A \rightarrow DB$ ,

$$\begin{array}{ccc}
 & \xrightarrow{\text{Id}_{DA}} & DA \\
 & \Downarrow \mathbb{D}_A & \\
 DA & \xrightarrow{dA^{\mathbb{D}}} & DA \\
 & \Downarrow \mathbb{D}_{dA, f} & \\
 & (f^{\mathbb{D}} dA)^{\mathbb{D}} & \\
 & \Downarrow (\mathbb{D}_f)^{\mathbb{D}} & \\
 & & DB = id_{f^{\mathbb{D}}} \\
 & \xrightarrow{f^{\mathbb{D}}} &
 \end{array}$$

# Pseudomonads

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 & \Downarrow \mathbb{D}_{dA} & \\
 & dA^{\mathbb{D}} & \\
 & \Downarrow \mathbb{D}_A & \\
 & dA & \\
 & \Downarrow \text{Id}_{DA} & \\
 & DA &
 \end{array}
 \quad \text{Id}_{DA} = id_{dA}$$

- For every  $f : A \rightarrow DB$ ,

$$\begin{array}{ccccc}
 & & \text{Id}_{DA} & \nearrow & DA \\
 & & \Downarrow \mathbb{D}_A & & \\
 DA & \xrightarrow{dA^{\mathbb{D}}} & DA & \xrightarrow{f^{\mathbb{D}}} & DB \\
 & \Downarrow \mathbb{D}_{dA, f} & & & \\
 & (f^{\mathbb{D}} dA)^{\mathbb{D}} & \Downarrow (\mathbb{D}_f)^{\mathbb{D}} & & \\
 & \searrow & \Downarrow & & \\
 & & f^{\mathbb{D}} & \searrow & \\
 & & & & DB = id_{f^{\mathbb{D}}}
 \end{array}$$

# Pseudomonads

- For every  $f : A \rightarrow DB$ ,

$$\begin{array}{ccc}
 & DB & \\
 f^{\mathbb{D}} \nearrow & & \searrow \text{Id}_{DB} \\
 DA & & DB \\
 \mathbb{D}_{f, dB} \swarrow & & \swarrow \mathbb{D}_B \\
 & dB^{\mathbb{D}} & \\
 \text{---} \searrow & & \text{---} \\
 & DB = (\mathbb{D}_B f)^{\mathbb{D}} & \\
 (dB^{\mathbb{D}} f)^{\mathbb{D}} \swarrow & & \swarrow
 \end{array}$$

- For every 2-cell  $\varphi : f \rightarrow g : A \rightarrow DB$ ,

$$\begin{array}{ccc}
 A & \xrightarrow{dA} & DA \\
 g \searrow & \mathbb{D}_g \swarrow & \downarrow \mathbb{D}_f \\
 & g^{\mathbb{D}} \swarrow & f^{\mathbb{D}} \\
 & \varphi^{\mathbb{D}} \swarrow & \\
 & DB & \\
 \text{---} \searrow & & \text{---} \\
 & DB & \\
 g \searrow & \mathbb{D}_g \swarrow & \downarrow \mathbb{D}_f \\
 & g^{\mathbb{D}} \swarrow & f^{\mathbb{D}} \\
 & \varphi \swarrow & \\
 & DB &
 \end{array}$$

# Pseudomonads

- For every  $f : A \rightarrow DB$ ,

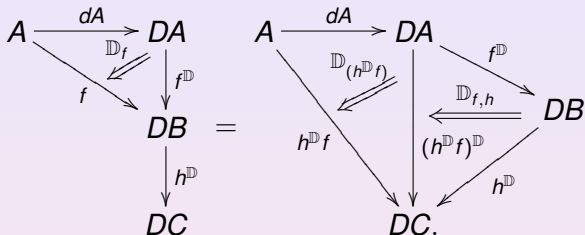
$$\begin{array}{ccc}
 & DB & \\
 f^{\mathbb{D}} \nearrow & & \searrow \text{Id}_{DB} \\
 DA & & DB \\
 \mathbb{D}_{f, dB} \swarrow & & \swarrow \mathbb{D}_B \\
 & dB^{\mathbb{D}} & \\
 \text{---} \searrow & & \text{---} \\
 & DB = (\mathbb{D}_B f)^{\mathbb{D}} & \\
 (dB^{\mathbb{D}} f)^{\mathbb{D}} \swarrow & & \swarrow
 \end{array}$$

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$$\begin{array}{ccc}
 A & \xrightarrow{dA} & DA \\
 g \searrow & \mathbb{D}_g \swarrow & \downarrow g^{\mathbb{D}} \\
 & & DB \\
 & \swarrow \varphi^{\mathbb{D}} & \swarrow f^{\mathbb{D}} \\
 & & DB
 \end{array}
 =
 \begin{array}{ccc}
 A & \xrightarrow{dA} & DA \\
 g \searrow & \mathbb{D}_f \swarrow & \downarrow f^{\mathbb{D}} \\
 & & DB \\
 & \swarrow \varphi & \swarrow f \\
 & & DB
 \end{array}$$

# Pseudomonads

- For every  $f : A \rightarrow DB$ ,  $h : B \rightarrow DC$



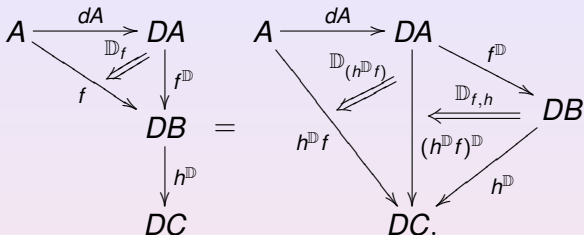
- For every  $\varphi : f \rightarrow g : A \rightarrow DB$  and every  $h : B \rightarrow DC$ ,



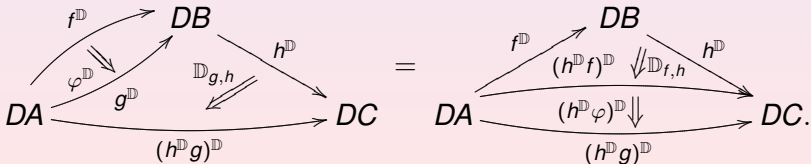


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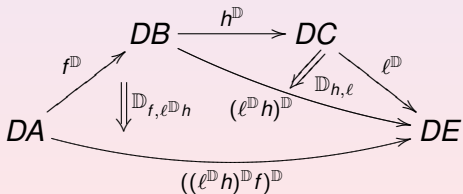
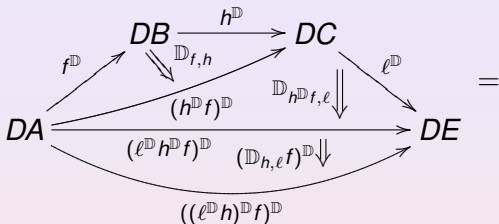
# Pseudomonads

- For every  $f : A \rightarrow DB$  and every  $\psi : h \rightarrow k : B \rightarrow DC$ ,

$$\begin{array}{ccc}
 & DB & \\
 f^{\mathbb{D}} \nearrow & & \searrow h^{\mathbb{D}} \\
 DA & & DC \\
 & \mathbb{D}_{f,k} \swarrow & \searrow k^{\mathbb{D}} \\
 & & \psi^{\mathbb{D}}
 \end{array}
 \quad = \quad
 \begin{array}{ccc}
 & DB & \\
 f^{\mathbb{D}} \nearrow & & \searrow h^{\mathbb{D}} \\
 DA & & DC \\
 & (h^{\mathbb{D}}f)^{\mathbb{D}} \searrow \Downarrow \mathbb{D}_{f,h} & \\
 & (\psi^{\mathbb{D}}f)^{\mathbb{D}} \Downarrow & \\
 & (k^{\mathbb{D}}f)^{\mathbb{D}} &
 \end{array}$$

# Pseudomonads

- For every  $f : A \rightarrow DB$ ,  $h : B \rightarrow DC$  and  $\ell : C \rightarrow DE$ ,



**Proposition.** We induce a pseudofunctor  $D: \mathcal{A} \rightarrow \mathcal{A}$ :  
 $D$  as given on objects.

For objects  $A, B$  we define  $D_{A,B}$  as the composite

$$\mathcal{A}(A, B) \xrightarrow{- \circ d_B} \mathcal{A}(A, DB) \xrightarrow{(\ )^{\mathbb{D}}} \mathcal{A}(DA, DB).$$

For every  $A$ ,  $D_A = \mathbb{D}_A$ .

For  $f: A \rightarrow B$  and  $h: B \rightarrow C$ , define

$D^{f,h}: Dh Df \rightarrow D(hf)$  as  $(\mathbb{D}_{d_C h} f)^{\mathbb{D}} \cdot \mathbb{D}_{d_B f, d_C h}$ .

# Pseudomonads

The rest of the structure for a usual pseudomonad is:  
For every  $f : A \rightarrow B$  we define  $d_f := \mathbb{D}_{dB} f$ :

$$\begin{array}{ccc} A & \xrightarrow{dA} & DA \\ f \downarrow & \swarrow d_f & \downarrow (dB f)^{\mathbb{D}} = Df \\ B & \xrightarrow{dB} & DB. \end{array}$$

# Pseudomonads

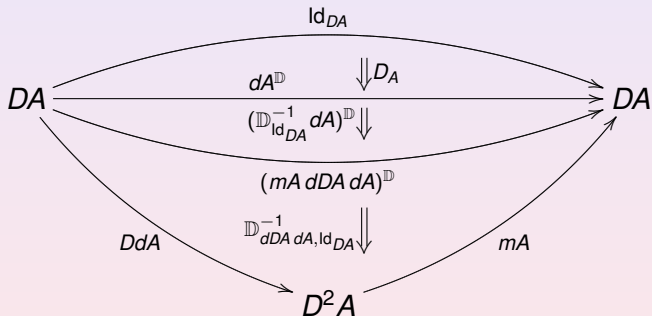
For every  $A$  we define  $m_A = (\text{Id}_{DA})^{\mathbb{D}}$  and for every  $f : A \rightarrow B$  we define  $m_f$  as

$$\begin{array}{ccc}
 D^2 A & \xrightarrow{(\text{Id}_{DA})^{\mathbb{D}}} & DA \\
 \downarrow D^2 f & \searrow (Df)^{\mathbb{D}} & \downarrow Df \\
 D^2 B & \xrightarrow{((\text{Id}_{DB})^{\mathbb{D}} dDB Df)^{\mathbb{D}}} & DB \\
 & \searrow ((\mathbb{D}_{dDB Df, \text{Id}_{DB}}^{-1})^{\mathbb{D}}) & \\
 & \searrow ((\mathbb{D}_{\text{Id}_{DB}}^{-1} Df)^{\mathbb{D}}) & \\
 & \searrow (\mathbb{D}_{\text{Id}_{DA}, dB f}) & 
 \end{array}$$

# Pseudomonads

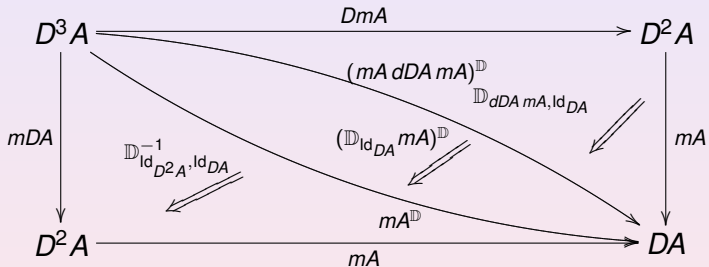
$$\beta A = \mathbb{D}_{\text{Id}_{DA}}$$

$\eta A$  as



# Pseudomonads

$\mu A$  is

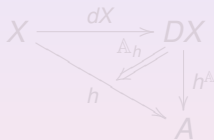




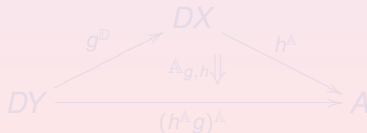
# An equivalent definition of the algebras

An algebra for the pseudomonad  $\mathbb{D}$  consists of

- An object  $A$  in  $\mathcal{A}$ .
- For every  $X$  in  $\mathcal{A}$  a functor  $(\ )^A : \mathcal{A}(X, A) \rightarrow \mathcal{A}(DX, A)$ .
- For every  $h : X \rightarrow A$  an invertible 2-cell



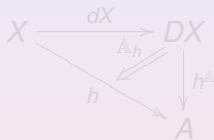
- For every  $h : X \rightarrow A$  and every  $g : Y \rightarrow DX$ , an invertible 2-cell



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$$\begin{array}{ccc} X & \xrightarrow{dX} & DX \\ & \searrow h & \downarrow h^{\mathbb{A}} \\ & & A \end{array}$$

$\mathbb{A}_h$  (2-cell between  $dX$  and  $h$ )

- For every  $h : X \rightarrow A$  and every  $g : Y \rightarrow DX$ , an invertible 2-cell

$$\begin{array}{ccc} & DX & \\ g^{\mathbb{D}} \nearrow & & \searrow h^{\mathbb{A}} \\ DY & \xrightarrow{\quad} & A \end{array}$$

$\mathbb{A}_{g,h}$  (2-cell between  $g^{\mathbb{D}}$  and  $h^{\mathbb{A}}$ )  
 $(h^{\mathbb{A}} g)^{\mathbb{A}}$  (bottom arrow)

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A commutative triangle diagram with vertices  $X$ ,  $DX$ , and  $A$ . The top edge is a horizontal arrow from  $X$  to  $DX$  labeled  $dX$ . The right edge is a vertical arrow from  $DX$  to  $A$  labeled  $h^{\mathbb{A}}$ . The bottom edge is a diagonal arrow from  $X$  to  $A$  labeled  $h$ . A double-headed arrow between the top and right edges is labeled  $\mathbb{A}_h$ .

- For every  $h : X \rightarrow A$  and every  $g : Y \rightarrow DX$ , an invertible 2-cell

A commutative triangle diagram with vertices  $DY$ ,  $DX$ , and  $A$ . The top-left edge is a diagonal arrow from  $DY$  to  $DX$  labeled  $g^{\mathbb{D}}$ . The top-right edge is a diagonal arrow from  $DX$  to  $A$  labeled  $h^{\mathbb{A}}$ . The bottom edge is a horizontal arrow from  $DY$  to  $A$  labeled  $(h^{\mathbb{A}}g)^{\mathbb{A}}$ . A double-headed arrow between the top and bottom edges is labeled  $\mathbb{A}_{g,h}$ .

# An equivalent definition of the algebras

Subject to the axioms

- For every  $\varphi : h \rightarrow k : X \rightarrow A$

$$\begin{array}{ccc}
 X & \xrightarrow{dX} & DX \\
 & \searrow \mathbb{A}_k & \swarrow \mathbb{A}_h \\
 & & A \\
 & \searrow k & \swarrow h^{\mathbb{A}} \\
 & & A
 \end{array}
 \quad
 \begin{array}{c}
 \left( \begin{array}{c}
 \mathbb{A}_k \circ \varphi^{\mathbb{A}} \\
 \varphi^{\mathbb{A}} \circ \mathbb{A}_h
 \end{array} \right)
 \end{array}
 \quad
 =
 \quad
 \begin{array}{ccc}
 X & \xrightarrow{dX} & DX \\
 & \searrow \varphi & \swarrow h \\
 & & A \\
 & \searrow k & \swarrow h^{\mathbb{A}} \\
 & & A
 \end{array}$$

- For every  $h : X \rightarrow A$

$$\begin{array}{ccc}
 & \xrightarrow{\text{Id}_{DX}} & DX \\
 & \searrow D_X & \swarrow h^{\mathbb{A}} \\
 DX & \xrightarrow{dX^{\mathbb{D}}} & A \\
 & \searrow (h^{\mathbb{A}} dX)^{\mathbb{A}} & \swarrow \text{id}_{h^{\mathbb{A}}} \\
 & & A
 \end{array}
 \quad
 \begin{array}{c}
 \left( \begin{array}{c}
 \mathbb{A}_{dX,h} \\
 \mathbb{A}_h^{\mathbb{A}}
 \end{array} \right)
 \end{array}$$

# An equivalent definition of the algebras

Subject to the axioms

- For every  $\varphi : h \rightarrow k : X \rightarrow A$

$$\begin{array}{ccc}
 X & \xrightarrow{dX} & DX \\
 \searrow & \swarrow \mathbb{A}_k & \downarrow h^A \\
 & & A \\
 \swarrow k & \nwarrow k^A & \uparrow \varphi \\
 & & 
 \end{array}
 =
 \begin{array}{ccc}
 X & \xrightarrow{dX} & DX \\
 \searrow & \swarrow \mathbb{A}_h & \downarrow h^A \\
 & & A \\
 \swarrow k & \nwarrow \varphi & \uparrow h^A \\
 & & 
 \end{array}$$

- For every  $h : X \rightarrow A$

$$\begin{array}{ccc}
 & \xrightarrow{\text{Id}_{DX}} & DX \\
 & \searrow D_X & \swarrow h^A \\
 DX & \xrightarrow{dX^{\mathbb{D}}} & A = \text{id}_{h^A} \\
 & \swarrow (h^A dX)^{\mathbb{A}} & \downarrow (\mathbb{A}_h)^{\mathbb{A}} \\
 & \searrow h^A & 
 \end{array}$$

# An equivalent definition of the algebras

- For every  $h: X \rightarrow A$  and every  $g: Y \rightarrow DX$

$$\begin{array}{ccc}
 Y & \xrightarrow{dY} & DY \\
 & \searrow g & \downarrow g^{\mathbb{D}} \\
 & & DX \\
 & & \downarrow h^A \\
 & & A
 \end{array}
 \quad = \quad
 \begin{array}{ccccc}
 Y & \xrightarrow{dY} & DY & & \\
 & \searrow h^A g & \downarrow & \searrow g^{\mathbb{D}} & \\
 & & A & & DX \\
 & & & \swarrow (h^A g)^A & \\
 & & & \swarrow \mathbb{A}_{g,h} & \\
 & & & & A
 \end{array}$$

# An equivalent definition of the algebras

- For every  $h: X \rightarrow A$  and every  $\psi: f \rightarrow g: Y \rightarrow DX$

$$\begin{array}{ccc}
 & & DX \\
 & \nearrow^{f^{\mathbb{D}}} & \\
 DY & \xrightarrow{\psi^{\mathbb{D}}} & \\
 & \searrow_{g^{\mathbb{D}}} & \\
 & & A
 \end{array}
 \begin{array}{c}
 \swarrow_{h^A} \\
 \downarrow_{\mathbb{A}_{g,h}} \\
 \swarrow_{(h^A g)^A}
 \end{array}
 =
 \begin{array}{ccc}
 & & DX \\
 & \nearrow^{f^{\mathbb{D}}} & \\
 DY & \xrightarrow{(h^A f)^A} & \\
 & \searrow_{(h^A \psi)^A} & \\
 & & A
 \end{array}
 \begin{array}{c}
 \swarrow_{h^A} \\
 \downarrow_{\mathbb{A}_{f,h}} \\
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 \end{array}$$

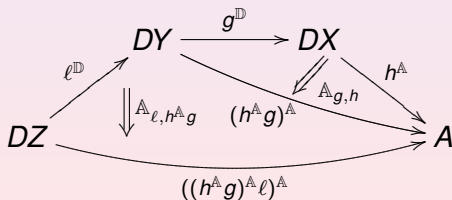
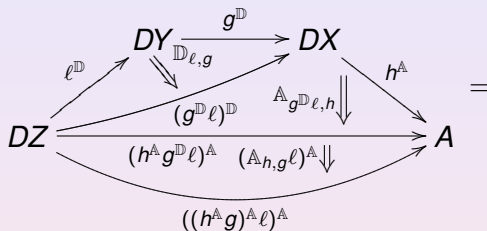
- For every  $\varphi: h \rightarrow k: X \rightarrow A$  and every  $g: Y \rightarrow DX$

$$\begin{array}{ccc}
 & & DX \\
 & \nearrow^{g^{\mathbb{D}}} & \\
 DY & \xrightarrow{\varphi^A} & \\
 & \searrow_{k^A} & \\
 & & A
 \end{array}
 \begin{array}{c}
 \swarrow_{h^A} \\
 \downarrow_{\mathbb{A}_{g,k}} \\
 \swarrow_{(k^A g)^A}
 \end{array}
 =
 \begin{array}{ccc}
 & & DX \\
 & \nearrow^{g^{\mathbb{D}}} & \\
 DY & \xrightarrow{(\varphi^A g)^A} & \\
 & \searrow_{h^A} & \\
 & & A
 \end{array}
 \begin{array}{c}
 \swarrow_{h^A} \\
 \downarrow_{\mathbb{A}_{g,h}} \\
 \swarrow_{(k^A g)^A}
 \end{array}$$



# An equivalent definition of the algebras

- For every  $h: X \rightarrow A$ ,  $g: Y \rightarrow DX$  and  $\ell: Z \rightarrow DY$



# An equivalent definition of the algebras

Given  $\mathbb{A} = (A, ( )^{\mathbb{A}})$  and  $\mathbb{B} = (B, ( )^{\mathbb{B}})$

a 1-cell  $\mathbb{A} \rightarrow \mathbb{B}$

is a 1-cell  $f : A \rightarrow B$  in  $\mathcal{A}$

together with an invertible 2-cell

$$\begin{array}{ccc} DX & & \\ h^A \downarrow & \searrow (fh)^{\mathbb{B}} & \\ A & \xrightarrow{f} & B \\ & \swarrow f[h] & \end{array}$$

for every  $h : X \rightarrow A$  in  $\mathcal{A}$ .

# An equivalent definition of the algebras

Given  $\mathbb{A} = (A, ( )^{\mathbb{A}})$  and  $\mathbb{B} = (B, ( )^{\mathbb{B}})$

a 1-cell  $\mathbb{A} \rightarrow \mathbb{B}$

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together with an invertible 2-cell

$$\begin{array}{ccc} DX & & \\ \downarrow h^{\mathbb{A}} & \searrow (fh)^{\mathbb{B}} & \\ A & \xrightarrow{f} & B \\ & \swarrow f[h] & \end{array}$$

for every  $h : X \rightarrow A$  in  $\mathcal{A}$ .

# An equivalent definition of the algebras

Subject to the coherence conditions

$$\begin{array}{ccc} X & \xrightarrow{dX} & DX \\ & \searrow h & \downarrow h^A \\ & & A \\ & & \xrightarrow{f} B \end{array} \quad \begin{array}{ccc} & \swarrow (fh)^B & \\ & & \downarrow (fh)^B \\ & & B \end{array} \quad = \quad \begin{array}{ccc} X & \xrightarrow{dX} & DX \\ & \downarrow h & \downarrow h^A \\ & & A \\ & & \xrightarrow{f} B \end{array} \quad \begin{array}{ccc} & \swarrow \mathbb{B}_{fh} & \\ & & \downarrow (fh)^B \\ & & B \end{array}$$

# An equivalent definition of the algebras

for any  $g: Y \rightarrow DX$

$$\begin{array}{ccc}
 DY & \xrightarrow{g^D} & DX \\
 \searrow & \swarrow \mathbb{A}_{g,h} & \downarrow h^A \\
 & & A \\
 \searrow (h^A g)^A & & \xrightarrow{f} B \\
 & & \swarrow f[h] \\
 & & (fh)^B
 \end{array}
 =$$

$$\begin{array}{ccc}
 DY & \xrightarrow{g^D} & DX \\
 \downarrow (h^A g)^A & \searrow & \downarrow (fh)^B \\
 A & \xrightarrow{f} & B \\
 \swarrow f[h^A g] & \searrow & \swarrow f \\
 & & (fh^B g)^B \\
 \swarrow & \searrow & \swarrow \\
 & & (f[h] g)^B \\
 \swarrow & \searrow & \swarrow \\
 & & ((fh)^B g)^B \\
 \swarrow & \searrow & \swarrow \\
 & & \mathbb{B}_{g, fh}
 \end{array}$$

# An equivalent definition of the algebras

and for every  $\kappa : h \rightarrow k : X \rightarrow A$ ,

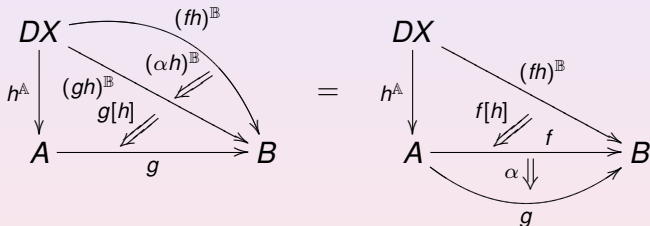
The diagram illustrates the equivalence of two definitions of an algebra. On the left, a commutative diagram shows a map  $DX \rightarrow A$  with two arrows:  $k^A$  (downward) and  $h^A$  (downward). A curved arrow  $\kappa^A$  connects  $h^A$  to  $k^A$ . A map  $A \rightarrow B$  is labeled  $f$ . A curved arrow  $(fh)^B$  goes from  $DX$  to  $B$ , and a straight arrow  $f[h]$  goes from  $DX$  to  $B$ . On the right, an equivalent diagram shows  $DX \rightarrow A$  with  $k^A$  (downward) and  $(fk)^B$  (downward). A curved arrow  $(f\kappa)^B$  connects  $(fk)^B$  to  $(fh)^B$ . A map  $A \rightarrow B$  is labeled  $f$ . A curved arrow  $(fh)^B$  goes from  $DX$  to  $B$ , and a straight arrow  $f[k]$  goes from  $DX$  to  $B$ . An equals sign  $=$  is placed between the two diagrams.

# An equivalent definition of the algebras

A 2-cell  $\alpha : f \rightarrow g : \mathbb{A} \rightarrow \mathbb{B}$

is a 2-cell  $\alpha : f \rightarrow g$

such that for every  $h : X \rightarrow A$  we have



# An equivalent definition of the algebras

**Teorema.** The usual bicategory of algebras for  $\mathbb{D}$  is equivalent to the bicategory of algebras just defined.