Hopf algebra generalisations

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CT2010

Outline

1. Hopf algebras
2. Finite Hopf algebras and Radford's formula
3. Pseudomonoids
4. Duals and dualizations
5. Radford's formula for autonomous pseudomonids
6. Recovering the classical Radford's formula
7. Radford's formula for autonomous categories

Hopf algebras

A Hopf algebra H is:

- H is a bialgebra
- $S : H \to H$ such that

$$(H \xrightarrow{\Delta} H^{\otimes 2} \xrightarrow{S\otimes 1} H^{\otimes 2} \xrightarrow{p} H) = (H \xrightarrow{\varepsilon} k \xrightarrow{j} H)$$
$$= (H \xrightarrow{\Delta} H^{\otimes 2} \xrightarrow{1\otimes S} H^{\otimes 2} \xrightarrow{p} H)$$

- Main property: $\mathbf{Comod}_f(H)$ is monoidal and has left duals.
- Examples:
  - (Quantum) Enveloping algebras of Lie algebras.
  - $k[G]$ for an algebraic group G.
    $S(f)(g) = f(g^{-1})$, for $f \in k[G], x \in G$.
- In general, $S^2 \neq 1$.

Finite Hopf algebras

Comod(H) for a *finite* Hopf algebra H has many similarities with $Rep(G)$ for a finite (or compact) group G.

- *Integrals:* An integral is a map $\phi : H \to k$ s.t. $(\phi \otimes 1)\Delta = \phi j$ (or in Sweedler's notaion $\sum \phi(x_1)x_2 = \phi(x)j$).
- *Unicity of integrals:* The space of integrals has dimension one. If ϕ is an integral and $\psi : H \to k$, then $\psi * \phi$ is again an integral. Then $\psi * \phi = \psi(a)\phi$ for a unique invertible group-like $a \in H$ (*modular element*).
- *Radford's formula:*

$$a^{-1}S^{-2}(x)a = S^2(\alpha \rightharpoonup x \leftharpoonup \alpha^{-1})$$

for all $x \in H$, where $\alpha : H \to k$ is "dual to a" and $\rightharpoonup, \leftharpoonup$ are actions of H^* on H.

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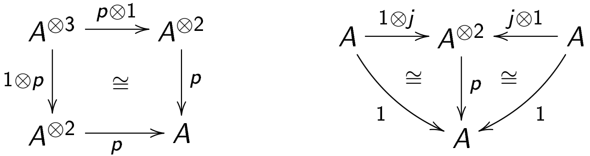
Pseudomonoids

A pseudomonoid in a monoidal bicategory $\mathcal{M}$ is an object A with

$$p : A \otimes A \to A \qquad j : I \to A$$



+ axioms

Example

- (Monoidal categories) When $(\mathcal{M}, I, \otimes) = (\mathbf{Cat}, \mathbf{1}, \times)$, pseudomonoid = monoidal category.
- (Coquasi bialgebras) When $(\mathcal{M}, I, \otimes) = (\mathbf{Comon}, k, \otimes)$ pseudomonoid = coquasi bialgebra (Drinfel'd).

Duals

Notation: if $A \in \mathrm{ob}\mathcal{M}$, denote a *right bidual* by A°, with evaluation and coevaluation $e : A \otimes A^\circ \to I$, $n : I \to A^\circ \otimes A$

Definition (Street-Wood)

Let A be a pseudomonoid in $\mathcal{M}$ and X an object with bidual. An *exact pairing* for $x : X \to A$, $y : X^\circ \to A$ has data + two axioms:

y is called a *left dual* of x.

- A left dual to 1_A is called a *left dualization*.

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\begin{array}{ccc}
X^\circ \otimes X & \xrightarrow{y \otimes x} & A^{\otimes 2} \\
{\scriptstyle n}\uparrow & \Downarrow \alpha & \downarrow{\scriptstyle p} \\
I & \xrightarrow[j]{} & A
\end{array}
\qquad
\begin{array}{ccc}
I & \xrightarrow{j} & A \\
{\scriptstyle e}\uparrow & \Downarrow \beta & \uparrow{\scriptstyle p} \\
X \otimes X^\circ & \xrightarrow[x \otimes y]{} & A^{\otimes 2}
\end{array}
$$

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Autonomous pseudomonoids

Definition (Day-McCrudden-Street)

- A left (right) autonomous pseudmonoid is one equipped with a left (right) dualization $d : A^\circ \to A$.
- An *automomous* pseudomonoid is one that is left and right autonomous.
- Obs: for any map $x : X \to A$, $x^+ = (X^\circ \xrightarrow{x^{*\circ}} A^\circ \xrightarrow{d} A)$

Now we assume $\mathcal{M}$ is braided with right (and then left) biduals. Let (A, j, p) be an *autonomous map* pseudomonoid in $\mathcal{M}$.

Definition

We say that $Z \in \mathrm{ob}\mathcal{M}$ is *finite* when $n : I \to Z^\circ \otimes Z$ is a map.

Notation: for $x : X \to A, y : Y \to A$,

$$x \bullet y : X \otimes Y \xrightarrow{x \otimes y} A \otimes A \xrightarrow{p} A$$

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Radford's formula

Theorem

Let A be a finite autonomous map pseudomonoid. For any map $x : X \to A$,

$$w^{-} \bullet x^{++} \bullet w \cong x^{--}$$

where $w : I \to A$ *is invertible in* $\mathcal{M}(I, A)$, *and defined by* $n^{-} \cong n \bullet w \in \mathcal{M}(I, A^{\circ} \otimes A)^{(n\bullet -)}$.

Key steps in the proof

- The monad $(n \bullet 1) : A^{\circ} \otimes A \to A^{\circ} \otimes A$ has E-M construction

$$A \xrightarrow{1\otimes n} A^{\circ} \otimes A \otimes A \xrightarrow{1\otimes p} A^{\circ} \otimes A$$

- $n^{-} \cong n \bullet w$ for a unique up to iso $w : I \to A$; w is a map.
- $n \bullet (z \otimes j) \cong n \bullet (j^{*\circ} \otimes dz)$ for any $z : Z \to A^{\circ}$.

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Example: comodules

There is a monoidal pseudofunctor **Comon** $\to$ **Comod**, where **Comod** is the bicategory of coalgebras, bicomodules and bicomodule morphisms.
An example of an autonomous pseudomonoid in **Comod** is a coquasi Hopf algebra; in particular a Hopf algebra.

Radford's formula

Let H be a (coquasi) Hopf algebra with multiplication $p : H^{\otimes 2} \to H$ and unit $j : k \to H$.

- H° is the opposite coalgebra and $n : k \to H^{\circ} \otimes H$ is H with a twisted regular structure.
- $n^{-} \cong n \bullet w$ becomes $H^{\vee} \cong H \otimes W$, and $W=$ integrals.
- Coaction $W \to W \otimes H$ corresponds to modular element $a \in H$.
- $w^{-} \bullet x^{++} \bullet w \cong x^{--}$ translates into $a^{-1} S^{-2}(x) a = S^{2}(\alpha \rightharpoonup x \leftharpoonup \alpha^{-1})$

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Autonomous categories

Let **Lex** be the 2-category of finitely complete **Vect**-categories and lex functors.

Lex is monoidal with

- tensor product $A \boxtimes B$ s.t. lex functors $A \boxtimes B \to C$ classify functors $A \otimes B \to C$ lex in each variable;
- unit object $\mathbf{Vect}_f^{op}$.

Example

Example of an autonomous map pseudomonoid in **Lex**: $\mathbf{Comod}_f(H)$ for H a finite Hopf algebra.

Radford's formula (Etingof-Nikshych-Ostrik)

If $\mathcal{C}$ is an autonomous **Vect**-category + finitness conditions,

$$W^* \otimes {}^{**}X \otimes W \cong X^{**}$$

for a canonical invertible object W and any X in $\mathcal{C}$.

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References

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