Span, Span, SpanSpan, SpanSpan

Toby Kenney
Joint with Dorette Pronk

Mathematics, Dalhousie University, Halifax, Canada

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Span as a profunctor

It is easy to compose a span with a morphism on the left:

$$A \xleftarrow{f} B \xrightarrow{g} C \xrightarrow{h} D$$

or a backwards morphism on the right:

$$X \xleftarrow{x} A \xleftarrow{f} B \xrightarrow{g} C$$

This makes Span into a profunctor $\mathcal{C}^{\mathrm{op}} \xrightarrow{\mathfrak{s}} C$.

Composition of Spans as a Natural Transformation

There is also a composable pair of spans profunctor, formed as the composite $\mathfrak{s}\mathfrak{c}\mathfrak{s}$, where $\mathfrak{s}$ is the span profunctor, and $\mathfrak{c}$ is the cospan profunctor.

The composition of spans is a natural transformation of profunctors $\mathfrak{s}\mathfrak{c}\mathfrak{s} \xrightarrow{\alpha} \mathfrak{s}$.

The Role of Pullbacks

Pullbacks provide a method for converting a cospan into a span. This can be described in terms of the natural transformation on the previous slide, as the application of the natural transformation to a composable pair of spans of the form

$$A = A \xrightarrow{f} B \xleftarrow{g} C = C$$

Conversely, given another appropriate method of converting cospans into spans, we can extend it uniquely to a natural transformation $\mathfrak{s}\mathfrak{c}\mathfrak{s} \xrightarrow{\alpha} \mathfrak{s}$.

Properties of Pullbacks

1. Given a pair of pullbacks:

$$\begin{array}{ccccc} Q & \xrightarrow{s} & P & \xrightarrow{q} & D \\ {\scriptstyle r}\downarrow & & {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle h} \\ A & \xrightarrow[f]{} & B & \xrightarrow[g]{} & C \end{array}$$

The composite

$$\begin{array}{ccc} Q & \xrightarrow{qs} & D \\ {\scriptstyle r}\downarrow & & \downarrow{\scriptstyle h} \\ A & \xrightarrow[gf]{} & C \end{array}$$

Is also a pullback.

Properties of Pullbacks II

2.

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ \| & & \| \\ A & \xrightarrow[f]{} & B \end{array}$$

Is a pullback.

We will choose a collection of squares with these properties, and call such squares canonical squares.

Examples

1. If $\mathcal{C}$ has functorial pullbacks, then we can choose canonical squares to be the given pullbacks.
2. For a monoid, M, we can take commutative squares of the form:

$$\begin{array}{ccc} \star & \xrightarrow{x} & \star \\ {\scriptstyle y}\downarrow & & \downarrow{\scriptstyle y} \\ \star & \xrightarrow[x]{} & \star \end{array}$$

Examples (Continued)

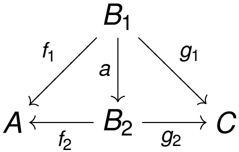
3. If $\mathcal{C}$ has an involution $(\_)^\circ$, then we can take canonical squares of the form

$$\begin{array}{ccc} A & \xrightarrow{g^\circ f} & C \\ {\scriptstyle 1_A}\downarrow & & \downarrow{\scriptstyle g} \\ A & \xrightarrow{f} & B \end{array}$$

However, these do not satisfy the unit laws. We will discuss how to deal with this later.

Two-cells Between Spans

There is a notion of 2-cell between spans, given by commutative diagrams of the form



Therefore, $\mathcal{S}pan(\mathcal{C})$ is usually studied as a bicategory. These 2-cells make $\mathfrak{s}$ into a $\mathcal{C}at$-valued profunctor.

Middle-4-interchange

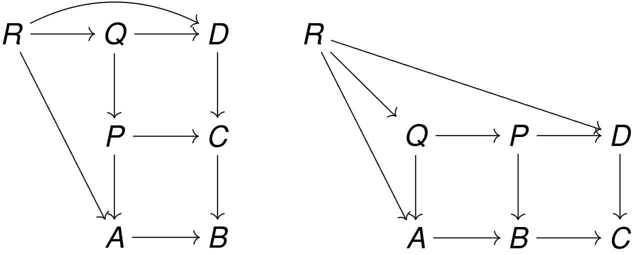
The middle-4-interchange law holds if and only if the canonical squares commute.

Isomorphisms

To get a bicategory, we only need α to be associative up to coherent isomorphism.

$$\alpha(\alpha\mathfrak{cs}) \xrightarrow{\theta} \alpha(\mathfrak{sc}\alpha)$$

This corresponds to a pair of isomorphisms between canonical squares:



Coherence Conditions

The standard coherence conditions for the associativity of α give rise to two sorts of coherence conditions for the isomorphisms between canonical squares:

- For three squares in a row horizontally or vertically:

$$\begin{array}{ccccccc} R & \longrightarrow & Q & \longrightarrow & P & \longrightarrow & E \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & D \end{array}$$

Coherence Conditions (continued)

- For a 2×2 grid of squares:

$$\begin{array}{ccccc} S & \longrightarrow & R & \longrightarrow & E \\ \downarrow & & \downarrow & & \downarrow \\ Q & \longrightarrow & P & \longrightarrow & D \\ \downarrow & & \downarrow & & \downarrow \\ A & \longrightarrow & B & \longrightarrow & C \end{array}$$

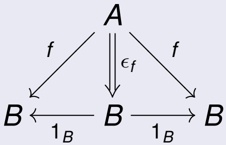
Universal Properties

Theorem

If the canonical squares of a morphism f with itself is reflexive, then $\mathcal{S}pan(\mathcal{C})$ is the initial bicategory with a sinister embedding of $\mathcal{C}$, subject to the Beck-Chevalley conditions for canonical squares.

Theorem

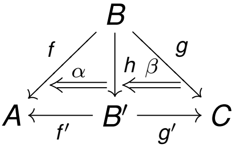
The bicategory of spans is the initial bicategory such that the image of every morphism has an opposite, with a counit



satisfying certain equations.

Span of a Bicategory

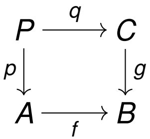
For Span of a bicategory, we define two-cells to be equivalence classes of diagrams of the form:



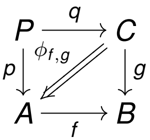
under the obvious equivalence relation.

Extra Conditions on the Canonical Squares

Any canonical square



must contain a chosen 2-cell, $\phi_{f,g}$.

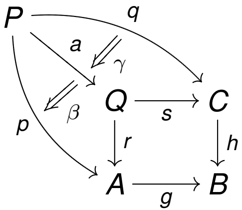


subject to the obvious composition rules.

Extra Conditions on Canonical Squares II

If
$$\begin{array}{ccc} P & \xrightarrow{q} & C \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle h} \\ A & \xrightarrow[f]{} & B \end{array} \qquad \text{and} \qquad \begin{array}{ccc} Q & \xrightarrow{s} & C \\ {\scriptstyle r}\downarrow & & \downarrow{\scriptstyle h} \\ A & \xrightarrow[g]{} & B \end{array}$$

are canonical squares, then for any 2-cell $f \xRightarrow{\alpha} g$, there is a morphism $P \xrightarrow{a} Q$, and a pair of 2-cells β and γ:

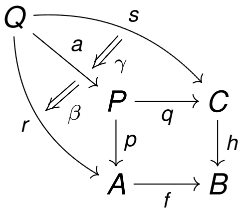


subject to some coherence and functoriality conditions.

Extra Conditions on Canonical Squares III

If
$$\begin{array}{ccc} P & \xrightarrow{q} & C \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle g} \\ A & \xrightarrow[f]{} & B \end{array} \qquad \text{and} \qquad \begin{array}{ccc} Q & \xrightarrow{s} & C \\ {\scriptstyle r}\downarrow & & \downarrow{\scriptstyle h} \\ A & \xrightarrow[f]{} & B \end{array}$$

are canonical squares, then for any 2-cell $g \overset{\alpha}{\Longrightarrow} h$, there is a morphism $Q \xrightarrow{a} P$, and a pair of 2-cells β and γ:



subject to some coherence and functoriality conditions.

When the Unit Law Fails

For many interesting examples of canonical squares, the unit law is not satisfied.

The unit law for canonical squares is what makes the identity span into an identity morphism in the category of spans.

Span of a Category Where Canonical Squares do not Satisfy the Unit Law

We form $\mathcal{S}pan(\mathcal{C})$ as the category generated by forwards and backwards morphisms from $\mathcal{C}$, subject to certain equations.

More precisely, we form the category $\mathcal{S}pan_0(\mathcal{C})$ freely generated by adjoining opposites to each morphism, subject to the Beck-Chevalley conditions.

Span for Canonical Squares that don't Satisfy the Unit Law

If

$$A \underset{g}{\overset{f}{\rightrightarrows}} B$$

are morphisms in $\mathcal{S}pan_0(\mathcal{C})$, and for any forward morphism $X \xrightarrow{a} A$ and any backward morphism $B \xleftarrow{b} Y$, the pairs of composites

$$X \xrightarrow{a} A \underset{g}{\overset{f}{\rightrightarrows}} B \qquad \text{and} \qquad A \underset{g}{\overset{f}{\rightrightarrows}} B \xleftarrow{b} Y$$

are both equal in $\mathcal{S}pan_0(\mathcal{C})$, then we identify f and g in $\mathcal{S}pan(\mathcal{C})$.

Canonical Squares in Span($\mathcal{C}$)

We can define canonical squares for $\mathcal{S}pan(\mathcal{C})$. If

$$\begin{array}{ccc} P & \xrightarrow{q} & D \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle h} \\ B & \xrightarrow[g]{} & C \end{array}$$

is a canonical square in $\mathcal{C}$, then we define

$$\begin{array}{ccccc} P & = & P & \xrightarrow{kq} & E \\ \| & & \| & & \uparrow{\scriptstyle k} \\ P & = & P & \xrightarrow[q]{} & D \\ {\scriptstyle fp}\downarrow & & \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle h} \\ A & \xleftarrow[f]{} & B & \xrightarrow[g]{} & C \end{array}$$

to be a canonical square in $\mathcal{S}pan(\mathcal{C})$.

Span as a monad

Using these canonical squares, we can iterate the span construction on categories with canonical squares.

There is an obvious functor $\mathcal{C} \longrightarrow \mathcal{S}pan(\mathcal{C})$, sending a morphism to the corresponding forward morphism in $\mathcal{S}pan(\mathcal{C})$.

There is a functor $\mathcal{S}pan(\mathcal{S}pan(\mathcal{C})) \longrightarrow \mathcal{S}pan(\mathcal{C})$ given by sending a pair of spans to the composite, replacing the backwards span with its opposite.

These make $\mathcal{S}pan$ into a monad on the bicategory of categories with canonical squares.

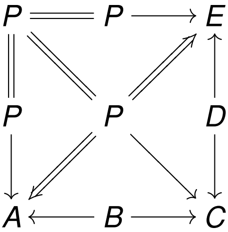
Algebras

Theorem

Algebras for the $\mathcal{S}pan$ monad are categories with canonical squares, with a contravariant identity-on-objects involution $(\_)^\circ$, satisfying the Beck-Chevalley condition for canonical squares.

Span of a Bicategory

On the other hand, for the bicategory $\mathcal{S}pan(\mathcal{C})$, the canonical square does not admit a single two-cell — it admits a span of two-cells:



Therefore, the span construction on bicategories cannot be iterated.

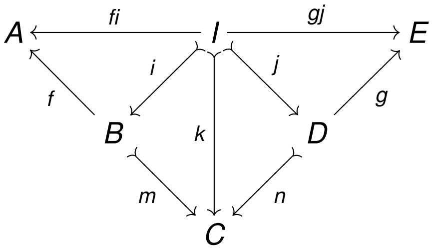
Not a Monad

It is not surprising that the span construction on a bicategory cannot be made into a monad on bicategories because it is not a completion — it adjoins an adjoint to each morphism, but not every morphism in the bicategory of spans has an adjoint.

$\mathcal{S}pan(\mathcal{P}art(\mathcal{C}))$

The canonical square for spans does contain a two-cell if either of the spans in it is a partial morphism. Therefore, we can form a bicategory $\mathcal{S}pan(\mathcal{P}art(\mathcal{C}))$.

We have a comparison morphism:

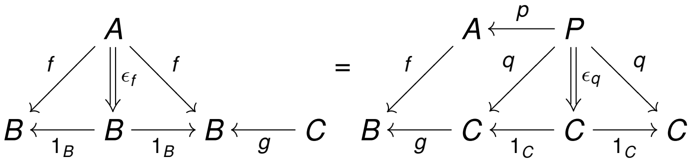


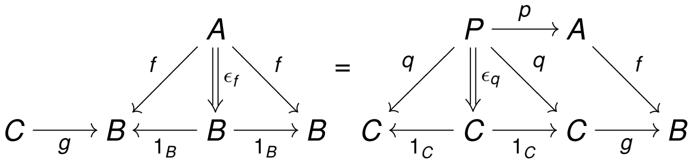
Thus the functor from $\mathcal{S}pan(\mathcal{P}art(\mathcal{C}))$ to $\mathcal{L}oc\mathcal{P}art(\mathcal{S}pan(\mathcal{C}))$ is in fact a right adjoint to the inclusion.

$\mathcal{P}art(\mathcal{P}art(\mathcal{C}))$

As a restriction of this, we can form the category $\mathcal{P}art(\mathcal{P}art(\mathcal{C}))$.
Doing this, we find that $\mathcal{P}art$ is a monad.
Algebras for the monad are bicategories in which every monomorphism has a right adjoint.
The partial map functor is a completion in this sense, because the right adjoints produced are not monomorphisms, so the new morphisms do not need adjoints adjoined to them.

Equations on Two-Cells in $Span(\mathcal{C})$

1. 

2. 

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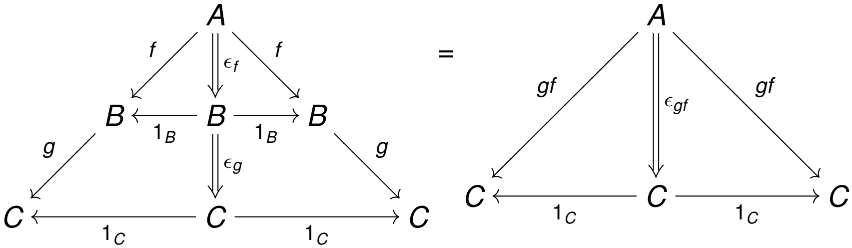
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