Kan extensions and reflections of presheaves preserving finite products

Panagis Karazeris
University of Patras
Patras, Greece

Grigoris Protsonis
University of Patras
Patras, Greece

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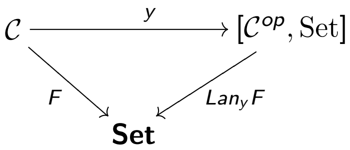

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$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{\quad y \quad} & [\mathcal{C}^{op}, \mathrm{Set}] \\ & {}_{F}\searrow \qquad \swarrow_{Lan_y F} & \\ & \mathbf{Set} & \end{array}$$

If we replace **Set** with a Grothendieck topos $\mathcal{E}$, then the cartesianess of $Lan_y F$ can be characterized by a similar condition, namely the filteredness of the dual of the category of elements of F in the internal logic of the topos (Diaconescu's theorem).

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• Good properties of the colimits used in the calculation of the Kan extension on the particular finite limits: The colimits have to behave "as in Sets" (but in a relative to a topology sense).

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This is not true:
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$$\begin{array}{ccc} \mathbf{BAlg_f} & \xrightarrow{\quad y \quad} & [\mathbf{BAlg_f}^{op}, \mathbf{Set}] \\ & {}_{i}\searrow \quad \swarrow_{Lan_y i} & \\ & \mathbf{BAlg} & \end{array}$$

i preserves finite limits, $\mathbf{BAlg_f}$ is dense in $\mathbf{BAlg}$ hence the "singular" functor $\mathbf{BAlg_f} \to [\mathbf{BAlg_f}^{op}, \mathbf{Set}]$ is fully faithful, and if $Lan_y i$ was cartesian, $\mathbf{BAlg}$ would have to be a topos. So the condition above is not sufficient.

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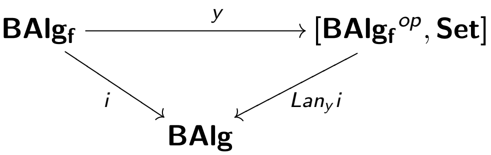

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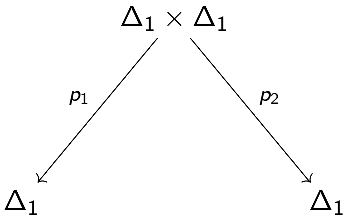

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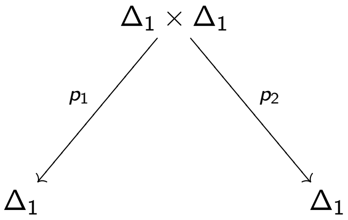

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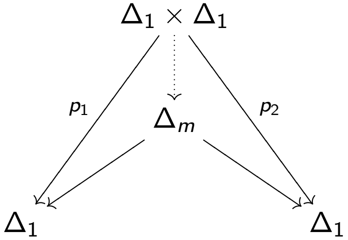

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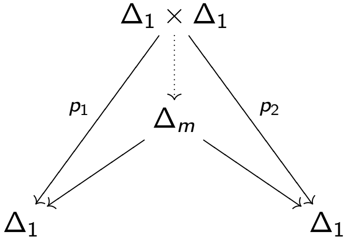

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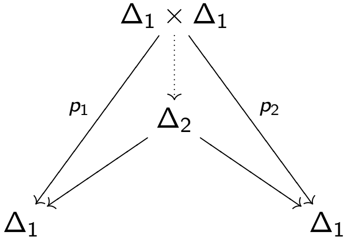

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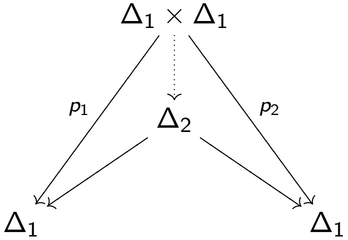

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Yet, $Lan_y \Delta$ is cartesian.

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The desired weakening of the notion of flatness has to do with the observation that the conditions for a functor $F : \mathcal{C} \longrightarrow \mathbf{Set}$, to be flat (i.e the dual of the category of elements of F to be filtered),can be interpreted in any category $\mathcal{E}$ where sheaf semantics are available. The general notion of flatness that prevails in the literature ($E \downarrow F$ is co-filtered) says that the category of elements of F is co-filtered with respect to the trivial topology on $\mathcal{E}$.

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(PC2) and if an element of the colimit has two representations as element of some components of the colimit, then there is a connecting zig-zag

$$\forall x : FC \ \forall y : FC'(l_C(x) = l_{C'}(y) \to \bigvee_{\mathcal{Z}(C,C')} \exists z_1 : FC_1 ... \exists z_n : FC_n$$

$$(Fd_{1,0}(z_1) = x \wedge Fd_{1,1}(z_1) = Fd_{2,0}(z_2) \wedge ... \wedge Fd_{n,1}(z_n) = y)),$$

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These are conditions that can be interpreted locally, with respect to some Grothendieck topology j on $\mathcal{E}$. We say that such a colimit is j-postulated.

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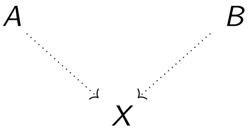
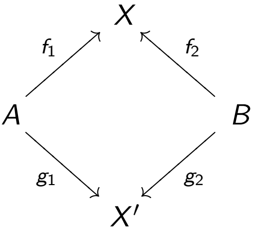
SF1. There exists at least one object $X \in \mathcal{C}$ and morphisms $A \to X$, $B \to X$

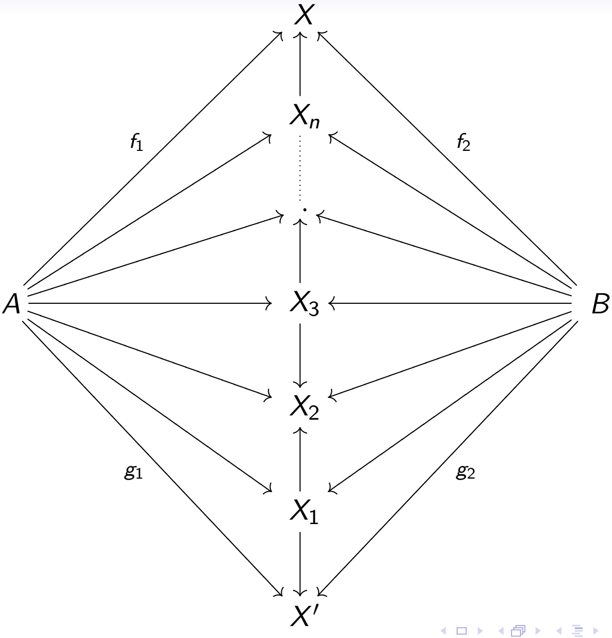


SF2. For each two objects in the category $(A, B) \downarrow \mathcal{C}$



there exists a zig-zag in this category connecting them.

Here is a zig-zag between co-spans!


X
X_n
f_1
f_2
A
X_3
B
X_2
g_1
g_2
X_1
X′

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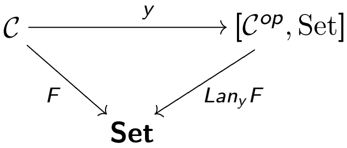

But (again, as in the case of flat functors) in general if $\mathcal{E}$ is a cocomplete category then the generalization of the above theorem isn't valid, if we stick to the absolute notion of "sifted flatness", i.e that for all $E \in \mathcal{E}$, $E \downarrow F$ is co-sifted.

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For this consider:

$$\begin{array}{ccc} \Delta & \xrightarrow{\quad y \quad} & [\Delta^{op}, \mathrm{Set}] \\ & {\scriptstyle U} \searrow \quad \swarrow {\scriptstyle Lan_y U} & \\ & \mathbf{Cat} & \end{array}$$

in which $Lan_y U$ preserves finite products, but U isn't sifted flat in this absolute sense (i.e with respect to the trivial topology).

Theorem

Let $F : \mathcal{C} \to (\mathcal{E}, j)$, where $(\mathcal{E}, j)$ is a loclly small, cocomplete, subcanonical site with finite products, be sifted flat (with respect to j). Assume that the colimit cocone $\{in_k : FP_k \to colimFP_k | \ k \in \mathcal{K}\}$, arising from the expression of a product of representables yC_1, yC_2 as a colimit of yP_k, satisfies PC1 (with respect to j). Then the morphism $f : colimFP_k \to FC_1 \times FC_2$ is an isomorphism. This means that $Lan_y F$ preserves the product $yC_1 \times yC_2$. Moreover, if $\mathcal{E}$ is cartesian closed, then $Lan_y F$ preserves finite products.

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Corollary

Let $F : \mathcal{C} \to \mathcal{E}$, where $\mathcal{E}$ is a Grothendieck topos. Then F is sifted flat in the internal logic, if and only if $Lan_y F$ preserves finite products.

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Using PC1 twice, for the morphisms $u : T \to colimFP_k$ and $v : T \to colimFP_k$, we have that there exists a cover $\{t_\alpha \colon T_\alpha \to T \mid \alpha \in A\}$ of T such that
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$$\begin{array}{ccc} T_\alpha & \xrightarrow{t_\alpha} & T \\ {\scriptstyle u_\alpha}\downarrow & & \downarrow{\scriptstyle u} \\ FP_k & \xrightarrow[in_k]{} & colimFP_k \end{array} \qquad \begin{array}{ccc} T_\alpha & \xrightarrow{t_\alpha} & T \\ {\scriptstyle v_\alpha}\downarrow & & \downarrow{\scriptstyle v} \\ FP_{k'} & \xrightarrow[in_{k'}]{} & colimFP_k \end{array}$$

The site is subcanonical, so it is enough to show that $u \circ t_\alpha = v \circ t_\alpha$ for each $\alpha \in A$, so from the above diagrams, we have to show $in_k \circ u_\alpha = in_{k'} \circ v_\alpha$, i.e that u_α and v_α "are equal in the colimit".

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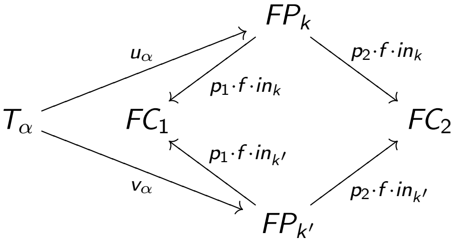

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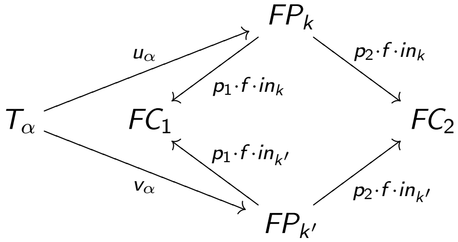
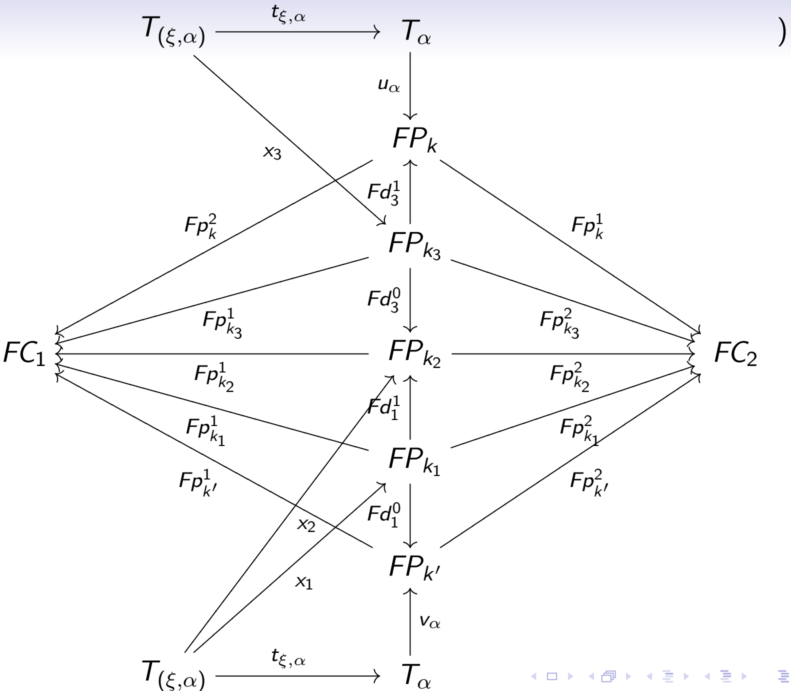


So, from SF2 (appropriately internalized) we have that there exists a zig-zag connecting P_k and $P_{k'}$, such that passing to $\mathcal{E}$ via F, certain equalities hold, upon replacing each T_α by a cover.


$T_{(\xi,\alpha)}$
$t_{\xi,\alpha}$
T_α
)
u_α
FP_k
x_3
Fd_3^1
Fp_k^2
Fp_k^1
FP_{k_3}
Fd_3^0
$Fp_{k_3}^1$
$Fp_{k_3}^2$
FC_1
FP_{k_2}
FC_2
$Fp_{k_2}^1$
$Fp_{k_2}^2$
Fd_1^1
$Fp_{k_1}^1$
$Fp_{k_1}^2$
FP_{k_1}
$Fp_{k'}^1$
$Fp_{k'}^2$
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(In the above diagram, for simplicity we assume that the zig-zag has length 3 and without loss of generality we can take the components of the zig-zag to be P_k's)
Those equalities tell us that $in_k \circ u_\alpha \circ t_{\xi,\alpha} = in_{k'} \circ v_\alpha \circ t_{\xi,\alpha}$ and from the fact that the site is subcanonical we have that $in_k \circ u_\alpha = in_{k'} \circ v_\alpha$.

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With similar techniques we prove that f is split epi (although for this we make no use of postulated colimits).

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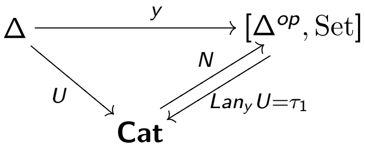

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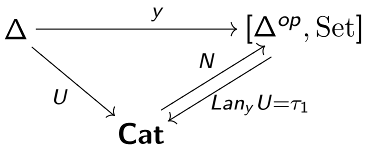

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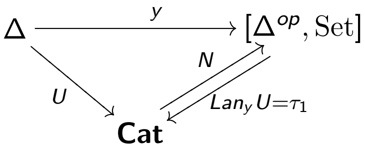

In any case, as an instance of the above results we have:
For any m and n, the product of representables $y[m] \times y[n] \cong \mathrm{colim}_i y[p_i]$ gives a postulated colimit $\mathrm{colim}_i Up_i$ with respect to the topology induced on **Cat** by that of epi-families on simplicial sets. This is essentially because this colimit is computed as in posets. (Only common parts of the $\binom{m+n}{n}$ paths of length $m+n$ are glued together, so there are no non-identity endomorphisms produced in the colimit.) Hence τ_1 preserves finite products.

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We might though be interested in more:

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$\mathcal{E}$ is extensive, regular epis are stable under pullbacks, countable coproducts are universal (but not disjoint!) ...
Idea of proof: The exactness conditions on $\mathcal{E}$ codify the postulatedness of the relevant colimits with respect to the topology induced on it by that of epi-families on the presheaf category. Flatness of the inclusion u w.r.t that topology is automatically satisfied.