ADVANCES IN SEMI-ABELIAN CATEGORICAL ALGEBRA

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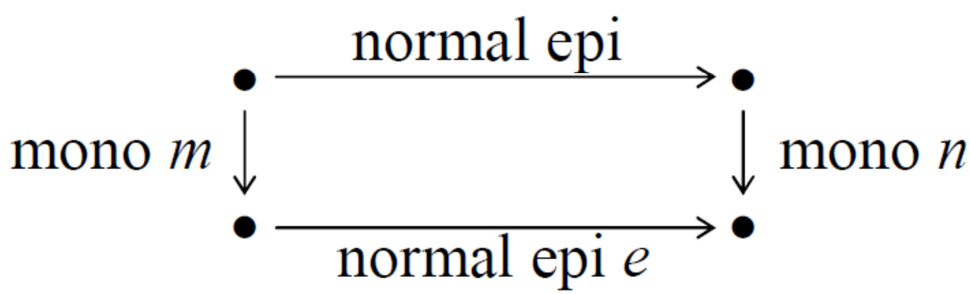
Abelian = Barr exact + Additive

What is the “non-abelian” version of this equation?

(The “non-abelian” is supposed to be to groups and rings as “abelian” is to abelian groups and modules)

Definition 1 ("new"). A pointed category **C** with finite limits and finite coproducts is said to be semi-abelian if it is Barr exact and *Bourn protomodular*. □

Definition 1 ("old"). A *normal* category **C** with finite coproducts is said to be semi-abelian if, given a commutative diagram in **C** of the form



we have:

(a) if m is a normal mono, then n is a normal mono;

(b) if n is a normal mono, and ker(e) factors through m, then m is a normal mono. □

Two discoveries of Dominique Bourn:

1. "Fibration of points": $\mathrm{Pt}_{\mathbf{C}}(B) = ((B,1_B)\downarrow(\mathbf{C}\downarrow B))$ = the category of triples (A,α,β), where $\alpha : A \to B$ and $\beta : B \to A$ are morphisms in $\mathbf{C}$ with $\alpha\beta = 1_B$.

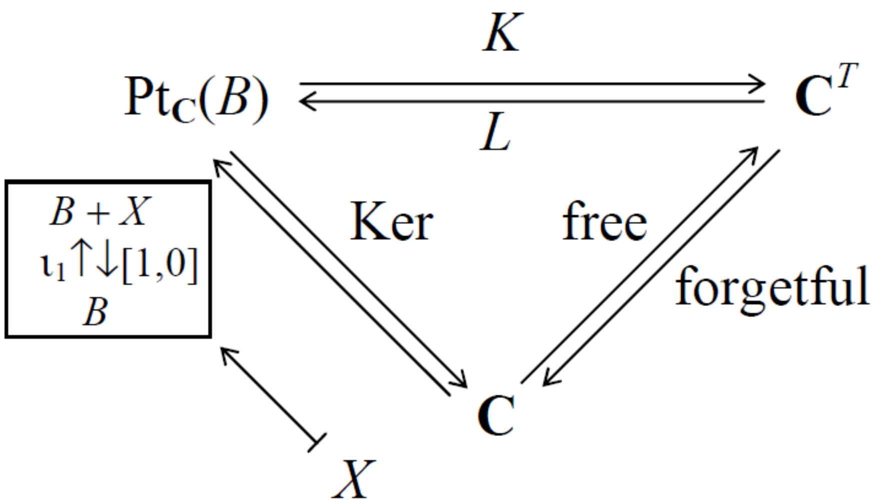
2. A category $\mathbf{C}$ is said to be (Bourn) *protomodular* if it has finite limits and, for every morphism $p : E \to B$ in $\mathbf{C}$, the pullback functor $p^* : \mathrm{Pt}_{\mathbf{C}}(B) \to \mathrm{Pt}_{\mathbf{C}}(E)$ reflects isomorphisms.

Two additional remarks:

1. If $\mathbf{C}$ is pointed (and has finite limits), then protomodularity is equivalent to *Split Short Five Lemma*.

2. If $\mathbf{C}$ is Barr exact, then Bourn protomodularity implies that each $p^* : \mathrm{Pt}_{\mathbf{C}}(B) \to \mathrm{Pt}_{\mathbf{C}}(E)$ is monadic.

Categorically defined semidirect products



When $E = 0$, the pullback functor $p^* : \mathrm{Pt}_{\mathbf{C}}(B) \to \mathrm{Pt}_{\mathbf{C}}(E)$ becomes the kernel functor $\mathrm{Ker} : \mathrm{Pt}_{\mathbf{C}}(B) \to \mathbf{C}$, sending (A,α,β) to the kernel $\mathrm{Ker}(\alpha)$. Therefore

$T(X) = B \flat X = \mathrm{Ker}(B+X \to B)$,

and, for a T-algebra (X,ξ), we define $B \ltimes (X,\xi) = L(X,\xi)$ via the coequalizer diagram

$$B \flat X \underset{\xi}{\overset{\mathrm{ker}}{\rightrightarrows}} B+X \longrightarrow L(X,\xi)$$

If $\mathbf{C} = \mathbf{Groups}$, then $\mathbf{C}^T = \mathbf{C}^B$ and $\xi(b,x,-b) = bx$. This suggests to write $\mathbf{C}^T = \mathbf{C}^B$ also for an arbitrary semi-abelian $\mathbf{C}$, and call T-algebra structures *internal B-actions*.

This makes Beck modules abelian objects in $\mathbf{C}^B$ *(as it should be!)*

Extensions with abelian kernels

$0 \to X \to ? \to B \to 0$ for an abelian X:

$$\text{“Ext”}(B,X) \overset{?}{=} \coprod_{\xi} \mathrm{Opext}(B,X,\xi), \quad \mathrm{Opext}(B,X,\xi) \approx \mathrm{H}^2(B,(X,\xi))$$

Disjoint union over all internal B-actions ξ on X

For groups, but later extended by J. Beck using appopriate H^1 instead of H^2, and replacing extensions with appropriate torsors.

Moore categories?: M. Gerstenhaber → D. Bourn → D. Rodelo
(but not all varieties of Ω-groups are Moore categories...)

Internal crossed modules

Definition 2. An internal crossed module in a semi-abelian category **C** is a system (B,X,ξ,f), in which (X,ξ) is an internal B-action and $f : X \to B$ a morphism in **C** making the diagrams

$$\begin{array}{ccc} B\flat X & \xrightarrow{\ker} & B+X \\ {\scriptstyle \xi}\downarrow & & \downarrow{\scriptstyle [1,f]} \\ X & \xrightarrow[f]{} & B \end{array} \qquad \begin{array}{ccc} (B+X)\flat X & \xrightarrow{[1,f]\flat 1} & B\flat X \\ {\scriptstyle \text{induced by}[1,\iota_2]}\downarrow & & \downarrow{\scriptstyle \xi} \\ B\flat X & \xrightarrow[\xi]{} & X \end{array}$$

commute. □

Question: Can the second diagram be replaced with

$$\begin{array}{ccc} X\flat X & \xrightarrow{f\flat 1} & B+X \\ {\scriptstyle \ker}\downarrow & & \downarrow{\scriptstyle \xi} \\ X+X & \xrightarrow[{[1,1]}]{} & X \end{array} \quad ?$$

Answer: If we do so, then the internal crossed modules will correspond to *star-multiplicative graphs*. Therefore this replacement can be made if and only if every internal star-multiplicative graph in **C** is an internal groupoid.

Commutator Theory 1: Motivation

Convention: all groups are additive

(1) The following conditions on a group G are equivalent:

(a) G admits an internal group structure;

(b) G admits a unique internal group structure;

(c) the addition map $G \times G \to G$ determines an internal group structure on G;

(d) the addition map $G \times G \to G$ is a homomorphism;

(e) G is abelian.

(2) More generally, for an internal reflexive graph $G = (G_0, G_1, d, c, e)$ in the category of groups the following conditions are equivalent:

(a) G admits an internal category structure;

(b) G admits a unique internal category structure;

(c) the map $G \times_{(d,c)} G \to G$ defined by $(f,g) \mapsto f - 1_{d(f)} + g$ determines an internal group structure on G;

(d) the map above is a homomorphism;

(e) $[\mathrm{Ker}(d), \mathrm{Ker}(c)] = 0$.

(3) Clearly (1) can be used to define the largest commutator, and then (2) can be used to define commutators of the form (H,K), where H and K are normal subgroups of a group G such that $G/H \approx G/K$ and the canonical map $G \to G/H$ is a split epimorphism. But what about the general case?

Commutator Theory 2: Definition

Definition 3. A (Kock) pregroupoid is a system $P = (P_0, P'_0, P_1, d, c, m)$, in which $P_0 \xleftarrow{d} P_1 \xrightarrow{c} P'_0$ is a span, and $m : P_1 \times_{(d,c)} P_1 \times_{(c,d)} P_1 \rightarrow P_1$ is map satisfying

$$dm(f,g,h) = d(h),\ cm(f,g,h) = c(f),\ \text{and}\ m(f,g,m(h,k,l)) = m(m(f,g,h),k,l),$$

whenever the terms involved are well defined. Internal pregroupoids in a category **C** with finite limits are defined accordingly. □

Consider the adjunction (whenever it exists):

$$\text{Span}(\mathbf{C}) \underset{U\,=\,\text{forgetful}}{\overset{F\,=\,\text{free}}{\rightleftarrows}} \text{Internal pregroupoids in } \mathbf{C},$$

and let us write $F(S_0 \xleftarrow{d} S_1 \xrightarrow{c} S'_0) = S_0 \xleftarrow{d} \overline{S_1} \xrightarrow{c} S'_0$.

Definition 4. Let $S = (S_0 \xleftarrow{d} S_1 \xrightarrow{c} S'_0)$ be a span in **C**. The commutator $[S]$ is the kernel pair of the canonical morphism $S_1 \rightarrow \overline{S_1}$. If R and R' are equivalence relations on an object A in **C**, and $S = (A/R \longleftarrow A \longrightarrow A/R')$, then we write $[S] = [R,R']$. □

Commutator Theory 3: Huq Commutator

Definition 5. The Huq commutator $[H,K]$ of normal subobjects $H = (H,h)$ and $K = (K,k)$ of an object A in **C** is the smallest normal subobject C of A, for which there exists a morphism $u : H{\times}K \to A/C$ making the diagram

$$
\begin{array}{ccccc}
 & H{+}K & \xrightarrow{[h,k]} & A & \\
\text{canonical morphism} & \downarrow & & \downarrow & \text{canonical morphism} \\
 & H{\times}K & \underset{u}{\dashrightarrow} & A/C &
\end{array}
$$

commute. □

The results

In a semi-abelian category **C**:

1. The Smith–Pedicchio commutator always exists (M. C. Pedicchio, 1995 or before, in a more general context).
2. Huq commutator always exists (D. Bourn, 2000?)
3. If **C** is strongly protomodular, then the two commutators always coincide (D. Bourn and M. Gran, 2000).
4. If the two commutators coincide, then every extension with an abelian kernel is a torsor (joint work with D. Bourn, published in 2004).
5. If the two commutators coincide, then every internal star-multiplicative graph is an internal groupoid, or, equivalently, internal crossed modules admit the simplified definition (N. Martins-Ferreira and T. Van der Linden, 2010).

More results very briefly:

1. Split Epimorphism Classifies and Borceux Representability Theorem.
2. Action Accessibility.
3. News from Milano, Calais, and Milano again: Sandra Mantovani and Giuseppe Metere; Andrea Montoli and Alan Cigoli on the action representability.
4. More commutators from Tomas Everaert and Tim Van der Linden.
5. More on Galois Theory, central and multiple central extensions, and Brown-Ellis-Hopf Formulae: Marino Gran, Tim Van der Linden, Tomas Everaert, Julia Goedecke.
6. On “semi-abelian and much more general”: many-many news, especially from Dominique Bourn, but also from the above-mentioned authors and others.

Algebraic exponentiation 1: We begin with:

A morphism $p : E \to B$ in $\mathbf{C}$, consider

$$p^* = p_a^* : \mathrm{Pt}_{\mathbf{C}}(B) \to \mathrm{Pt}_{\mathbf{C}}(E) \text{ versus } p^* = p_g^* : (\mathbf{C}\downarrow B) \to (\mathbf{C}\downarrow E),$$

and ask if these functors have right adjoints, and what are those? We observe:

1. For p_g^* the answers are well-known, and in particular:
   1.1. p_g^* has a right adjoint iff p is exponentiable.
   1.2. p_g^* has a right adjoint for each p iff $\mathbf{C}$ is locally cartesian closed.
   1.3. If $\mathbf{C}$ = **Sets**, then the pullback functor p_g^* "becomes" the composition-with-p functor $\mathbf{C}^p : \mathbf{C}^B \to \mathbf{C}^E$, and so it has both adjoints given by the Kan extensions along p.
2. The existence of the right adjoint for p_g^* easily implies the existence of the right adjoint for p_a^*.

Algebraic exponentiation 2: And it turns out that:

1. If **C** is semi-abelian, then p_g* : (**C**↓*B*) → (**C**↓*E*) exists *only* when *p* is an isomorphism (joint work with M. M. Clementino and D. Hofmann).
2. However, even if **C** is semi-abelian, p_a* : $\mathrm{Pt}_{\mathbf{C}}(B) \to \mathrm{Pt}_{\mathbf{C}}(E)$ might have a right for *every p*:
   - 2.1. If **C** is abelian, then simply because every p_a* is an equivalence.
   - 2.2. If **C** = **Groups**, then p_a* has a right adjoint for *the same reason* as why p_g* has right adjoint for **C** = **Sets**: indeed, if **C** = **Groups**, then p_g* "becomes" the composition-with-*p* functor $\mathbf{C}^p : \mathbf{C}^B \to \mathbf{C}^E$, and so its adjoints are again the Kan extensions along *p*.
3. That is, the category of groups being "opposite extreme to cartesian closed" nevertheless has its own "algebraic exponentiation", and the same is true for internal groups in a cartesian closed category.

Algebraic exponentiation 3: Additional remarks:

1. Two kinds of internal exponentiation for groups.
2. From groups to crossed modules and other internal categorical structures.
3. A "non-analogy": The "main" geometric and algebraic cases of exponentiability have $p : E \to 1$ and $p : 0 \to B$ respectively, yielding $T(D,d) = (E\times D,\mathrm{pr}_1)$ for the corresponding monad T in the first case, and $H(X) = B\times X$ for the corresponding comonad H in the second case for **C** = **Groups**. However, the appearances of $\times$ in both cases is only an "accident", not a sign of any analogy.
4. A new approach to semidirect products: When $p_a{}^*$ has a right adjoint $p_* : \mathrm{Pt}_{\mathbf{C}}(B) \to \mathrm{Pt}_{\mathbf{C}}(E)$, the adjunction is comonadic. We then take again $E = 0$, and the coalgebra structures become morphisms of the form $\xi : X \to H(X)$, which is nice, since $H(X) = B\times X$ in the case of groups, but "bad" for another reason.
5. A third kind of exponentiation in commutative algebra.

Two remarks on $\mathbb{Z}\flat\mathbb{Z} \to \mathbb{Z}+\mathbb{Z} \to \mathbb{Z}$:

1. Since the internal object actions in the category of groups are the same as the ordinary group actions, the corresponding monads are isomorphic, and so $\mathbb{Z}\flat\mathbb{Z} \approx \mathbb{Z}\cdot\mathbb{Z}$. This explains why every countable free group can be embedded into a free group on two generators.
2. The embedding $i : \mathbb{Z}\cdot\mathbb{Z} \approx \mathbb{Z}\flat\mathbb{Z} \to \mathbb{Z}+\mathbb{Z}$ can be described as follows:

- $\mathbb{Z}\cdot\mathbb{Z}$ is the free group on $\{x_0, x_1, x_2,\ldots\}$;
- $\mathbb{Z}+\mathbb{Z}$ is the free group on $\{y, z\}$;
- $i(x_n) = ny + z - ny$.

Then, for any non-trivial group G, let $f : \mathbb{Z}\cdot\mathbb{Z} \to G$ be the homomorphism having $f(x_0) = 0$ and $f(x_n) \neq 0$ for all $n > 0$. Since f cannot be extended to $\mathbb{Z}+\mathbb{Z}$, we conclude that G cannot be injective.