Iterating the icon construction

Nick Gurski, University of Sheffield

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The goal of this talk is to show how the construction of icons can be carried out in a very general setting, and in particular can be iterated to give a nice account of the structures involved at one stage of the Stabilization Hypothesis. This is joint work with Eugenia Cheng.

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Outline

1. Motivation: The Stabilization Hypothesis
   - Statement
   - Problems
2. Icons
   - The definition
   - Generalizing icons
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The Stabilization Hypothesis of Baez and Dolan relates n-categories with extra structure to certain special kinds of $(n+k)$-categories. The objects of interest are:

1. n-categories with k different, but compatible, multiplicative structures;
2. k-tuply monoidal n-categories, which are monoidal n-categories whose monoidal structure is more commutative the higher k is in comparison to n; and
3. k-degenerate $(n+k)$-categories, which are $(n+k)$-categories with only a single 0-cell, single 1-cell, and so on, up to a single $(k-1)$-cell.

$1 \simeq 2$ Categorified Eckmann-Hilton argument

$1 \simeq 3$ Straightforward

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There are three main problems one encounters in trying to make these comparisons precise.

- The dimensions of the total structures involved don't always agree.
- k-degenerate $(n+k)$-categories have a lot of extraneous data that doesn't appear in the other structures (e.g., associativity when $k=2$).
- The higher cells have extra data, and thus are the "wrong" things to use.

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Solution: build everything up at the correct dimension instead of trying to reduce dimensions later.

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Enrich in	**Cat**
0	Bicats
1	Functors
2	Icons
Total	**Icon**
Degen	1-degen = Mon. cats

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Enrich in	**Cat**	$\mathcal{K}$
0	Bicats	$\mathcal{K}$-cats
1	Functors	$\mathcal{K}$-functors
2	Icons	$\mathcal{K}$-icons
Total	**Icon**	$\mathcal{K}$-**Icon**
Degen	1-degen = Mon. cats	1-degen = $\mathcal{K}$-monoids

Solution: build everything up at the correct dimension instead of trying to reduce dimensions later.

Enrich in	**Cat**	$\mathcal{K}$	**Icon**
0	Bicats	$\mathcal{K}$-cats	Tricats*
1	Functors	$\mathcal{K}$-functors	Functors*
2	Icons	$\mathcal{K}$-icons	2x iconic trans
Total	**Icon**	$\mathcal{K}$-**Icon**	**2Icon**
Degen	1-degen = Mon. cats	1-degen = $\mathcal{K}$-monoids	2-degen = Braided

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Enrich in	**Cat**	$\mathcal{K}$	**Icon**	**nIcon**
0	Bicats	$\mathcal{K}$-cats	Tricats*	$(n+2)$-cats*
1	Functors	$\mathcal{K}$-functors	Functors*	Functors*
2	Icons	$\mathcal{K}$-icons	2x iconic trans	Higher icons
Total	**Icon**	$\mathcal{K}$-**Icon**	**2Icon**	$(\mathbf{n}+\mathbf{1})$**Icon**
Degen	1-degen = Mon. cats	1-degen = $\mathcal{K}$-monoids	2-degen = Braided	n-degen = Symmetric

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The key example: monoidal transformations We will consider the case of 1-degenerate 2-categories, or bicategories with a single object.

- 1-degenerate 2-categories $\leftrightarrow$ monoidal categories
- Functors between them $\leftrightarrow$ monoidal functors
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A pseudo-natural transformation $\alpha : F \Rightarrow G$ consists of

- for each 0-cell a in the source, a 1-cell $\alpha_a : Fa \to Ga$ in the target, and
- for each 1-cell $r : a \to b$ in the source, an invertible 2-cell

in the target, subject to two axioms.

In the degenerate case, this gives an object α_* and an isomorphism $\alpha_x : Gx \otimes \alpha_* \cong \alpha_* \otimes Fx$.

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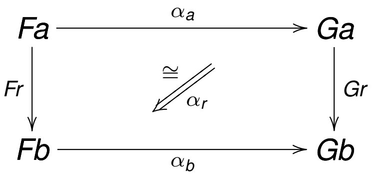

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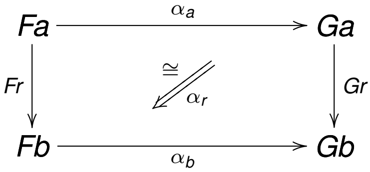

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To get a monoidal transformation from this data, we have to make 3 changes:

1. Remove the distinguished object α_*,
2. Remove the invertibility requirement on α_x, and
3. Reverse the direction of α_x.

Lack calls such a 2-cell an *icon*: an **I**dentity **C**omponent **O**plax **N**atural transformation.

To get a monoidal transformation from this data, we have to make 3 changes:

1. Remove the distinguished object α_*,
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Theorem.

- There is a 2-category of bicategories, weak functors, and icons which we will denote **Icon**.
- There is also a 2-category of category-enriched graphs, **Cat-Grph**, and a forgetful functor **Icon** $\rightarrow$ **Cat-Grph** which is monadic.
- The 2-category of monoidal categories is biequivalent to the full sub-2-category of **Icon** consisting of those objects with a single 0-cell.

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The abstract view of icons The data and axioms in the definition of an icon arise from two sources.

- The 2-cells in **Cat-Grph**
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What features of **Cat** have we used so far?

- The "free bicategory" 2-monad on **Cat-Grph**
  - Monoidal structure
  - Colimits preserved by the tensor product

Replace **Cat** with another bicategory $\mathcal{K}$ having the same features.

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$\mathcal{K}$-**Grph** A $\mathcal{K}$-graph X consists of

- a set of objects, X_0, and
- for each pair of objects $a, b \in X_0$, an object $X(a, b)$ of $\mathcal{K}$.

A map of $\mathcal{K}$-graphs $F : X \to Y$ consists of

- a function $F : X_0 \to Y_0$, and
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A graph 2-cell $\alpha : F \Rightarrow G$

- exists only when $Fa = Ga$ for all objects $a \in X$, and
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On $\mathcal{K}$-**Grph**, there is a monad T which is the "free category weakly enriched in $\mathcal{K}$" monad:

- $(TX)_0 = X_0$, so TX has the same objects, and
- $TX(a, b)$ is given by the coproduct

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The algebras for T are what you get if you take the definition of a bicategory and replace all the hom-categories with hom-objects from the monoidal bicategory $\mathcal{K}$.

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Examples of T-algebras

- $\mathcal{K} = \mathbf{Cat}$ gives bicategories
- $\mathcal{K} = \mathbf{Cat}_{\text{structured}}$ gives locally structured bicategories
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The total structure of T-algebras also has

- algebra maps $\leftrightarrow$ functors between bicategories
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- Compare to $\mathcal{V} \mapsto \mathcal{V}$-**Cat** for a monoidal category $\mathcal{V}$
- To form $\mathcal{V}$-**2Cat** $= (\mathcal{V}$-**Cat**$)$-**Cat**, we require a braiding on $\mathcal{V}$
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Now that I have outlined the general context in which we can form a bicategory of weakly enriched categories, functors, and icons, I would like to give some examples of these structures.

$\mathcal{K} =$ **Cat** is the relevant case for studying the Stabilization Hypothesis.

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A 0-cell X of **2Icon** has

- a set of objects, X_0;
- for each pair of objects, a bicategory $X(a, b)$;
- composition, which is a weak functor $\otimes : X(b, c) \times X(a, b) \to X(a, c)$;
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- associativity, and left and right unit icons, all subject to two axioms.

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- New feature: associativity and unit icons
- 1-cell composition is strictly associative and unital:

$$(f \otimes g) \otimes h = f \otimes (g \otimes h)$$
$$f \otimes I = f = I \otimes f$$

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- **Gray**-categories are special kinds of these

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- **Bicat**
- $\mathbf{Top}_3$
- In fact, $\Pi_2 : \mathbf{Top}_3 \to \mathbf{Bicat}$ is a 1-cell in **2Icon**.

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Proposition. The bicategory **PsMon**($\mathcal{K}$) of pseudomonoids in $\mathcal{K}$ is biequivalent to the full sub-bicategory of $\mathcal{K}$-**Icon** consisting of those 0-cells with a single object.

Theorem.

- The sub-2-category of **2Icon** consisting of the doubly degenerate objects is biequivalent to the 2-category **BrMonCat** of braided monoidal categories.
- For any $n \geq 3$, the sub-2-category of **nIcon** consisting of the n-denegenerate objects is biequivalent to the 2-category **SymMonCat** of symmetric monoidal categories.

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The general case

Enrich in	
0	
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Enrich in	**nCat**=**Cat**(n,0)
0	$(n+1)$-cats
1	Functors
2	"Icons"
⋮	⋮
$n+1$	$(n+1)$-cells*
Total	**Cat**(n,1)
Degen	1-degen = Mon. n-cats

The general case

Enrich in	**nCat**=**Cat**(n,0)	**Cat**(n,1)
0	$(n+1)$-cats	$(n+2)$-cats*
1	Functors	Functors*
2	"Icons"	Trans*
⋮	⋮	⋮
$n+1$	$(n+1)$-cells*	$(n+1)$-cells*
Total	**Cat**(n,1)	**Cat**(n,2)
Degen	1-degen = Mon. *n*-cats	2-degen = Braided *n*-cats

The general case

Enrich in	**nCat**=**Cat**(n,0)	**Cat**(n,1)	**Cat**(n,k-1)
0	$(n+1)$-cats	$(n+2)$-cats*	$(n+k)$-cats*
1	Functors	Functors*	Functors*
2	“Icons”	Trans*	Trans*
⋮	⋮	⋮	⋮
$n+1$	$(n+1)$-cells*	$(n+1)$-cells*	$(n+1)$-cells*
Total	**Cat**(n,1)	**Cat**(n,2)	**Cat**(n,k)
Degen	1-degen = Mon. *n*-cats	2-degen = Braided *n*-cats	*k*-degen = *k*-tuply monoidal