Structures as regimes in algebraic universes

A pulsation between discursive and visual worlds

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outline

Summary and References

Summary

In any *algebraic universe* $\mathcal{E}$ — and a fortiori in any topos or any category of fuzzy sets — a *N-regime of assimilations* is a N-dimensional thing, as is a N-category ; exactly it is in $\mathcal{E}$ a data

$$R = \left(a_N : E_N \to \mathcal{P}(E_{N-1}^2); a_{N-1} : E_{N-1} \to \mathcal{P}(E_{N-2}^2); \ldots, ; a_1 : E_1 \to \mathcal{P}(E_0^2) \right),$$

where at each level $k = 1, \ldots, N-1$ we assume the *transition axiom* (axiome de passage) :

$$\forall x, y \in E_k, \forall i \in E_{k+1} \Big[(y,x) \in a_k(i) \Rightarrow \exists r \in E_{k+2} \forall j \in E_{k+1} \big[(j,i) \in a_{k+1}(r) \to (y,x) \in a_k(j) \big] \Big].$$

The data of any usual structure in $\mathcal{E}$ could be presented as a N-regime of assimilations, and so through their underlying associated regimes, two heterogeneous structures (e.g. a group and a topology) could be put in communications, in the category $\mathcal{R}_N(\mathcal{E})$ of N-regimes in $\mathcal{E}$.
In fact a regime could be understood as an elementary calculus of variations, and this point of view allows us to construct a mathematical pulsation between the *discursive* world of modalities and the *visual* world of the morphological analysis. In particular the visual intuitions as well as the discursive concepts could be transposed in any regime.

References

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[3] Liftings of Stone's monadicity to spaces and the duality between the calculi of inverse and direct images, *Cahiers Top. Géo. Diff. Cat.* XL-2-2 (1999), p. 141-157.

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Algebraic universes

Definition of an algebraic universe

An *algebraic universe* (as developped in the 70's) is a category $\mathcal{C}$ with finite limits and colimits equipped with a contravariant functor $C : \mathcal{C}^{\mathrm{op}} \to \mathcal{C}$ (read Cf as "converse of f" or f^*) such that $C^{\mathrm{op}} \dashv C$, such that this adjonction is monadic (analogous to Stone duality), and such that, with $C(X) = \mathcal{P}(X)$, the canonical map $\eta_X : X \to \mathcal{P}^2(X)$ is factorized as $\eta_x = \psi_X.\alpha_X$, with $\alpha_X : X \to \mathcal{P}(X)$, $\psi_X : \mathcal{P}(X) \to \mathcal{P}^2(X)$ (meeting map) ; there are also $\pi_X : \mathcal{P}(X) \to \mathcal{P}^2(X)$ (inclusion map), $\nu_X : \mathcal{P}(X) \to \mathcal{P}(X)$ (negation), $c_X : X^2 \to \mathcal{P}(X)$ (pairing). Among these data we assume a precise system of equations, which provide *cartesian involutive monads*, etc. All that was proposed in order to construct equationally the theory of *continuous relations*.

Basic result : *In an* algebraic universe *any structure (in the Bourbaki's sense) could be defined equationally*.

Examples : the case of Ens or a topos

A topos $\mathcal{E}$ is an algebraic universe with $\mathcal{P}(X) = \Omega^X$, $\alpha_X(x) = \{x\}$, and
$\psi_X : \mathcal{P}(X) \to \mathcal{P}^2(X) : A \mapsto \{B; \exists x(x \in A \,\&\, x \in B)\}$,
$\pi_X : \mathcal{P}(X) \to \mathcal{P}^2(X) : A \mapsto \{B; \forall x(x \in B \Rightarrow x \in A)\}$.

(compare with the forthcoming definitions of $\Uparrow$ and of $\Downarrow$).

In any algebraic universe $\mathcal{P}$ is extended in two ways in a covariant functor by :
$\exists f = C(Cf.\alpha).\psi, \quad \forall f = C(Cf.\alpha).\pi.$

Notation : We often denote $\exists f$ by $\mathcal{P}(f)$.

Enlargement of logic by complete abelian monoïds

In a given algebraic universe, we get a covariant powerset monad $\mathbb{P} = (\mathcal{P}, \alpha, \cup)$, and if (A, θ, k) is a $\mathbb{P}$-algebra equipped with a compatible abelian monoïd law k (i.e. an abelian $\mathbb{L}$-algebra) we can construct in the same universe a new "fuzzy" logic for "fuzzy" relations (in the place of $\mathbb{P}$), with, in the place of $\mathcal{P}(X)$, the data $\mathcal{P}_A(X) = A^X$ (the exponentiation could be constructed).

THE FUNCTORS $\mathcal{L}$ AND $\mathcal{R}$

In an algebraic universe, if the sum $X^* = \sum_n X^n$ exists, then there is a distributive law from the monad $(-)^*$ toward the monad $\mathbb{P}$, and the composition monad $\mathbb{L}$ is the monad of languages, with endofunctor $\mathcal{L}(X) = \mathcal{P}(X^*)$.

The functor $\mathcal{R}(X) = \mathcal{P}(X^2)$, basic for the forthcoming question of regimes, is equipped with embeddings
$\mathcal{R}(X) = \mathcal{P}(X^2) \to \mathcal{P}(X^*) = \mathcal{L}(X)$.

In fact, as it is the case for $\mathcal{P}$, the functors $\mathcal{L}$ and $\mathcal{R}$ are also contravariant functors, and because of that several possibility exist for the definitions of morphisms of regimes, as it is the case for morphisms of spaces [which are not discussed in this talk]

Topology in algebraic universes and a duality principle

Interior and adherence, in mirror

If we define
$\Omega_X = \psi_{\mathcal{P}X} \circ \pi_X : \mathcal{P}(X) \to \mathcal{P}^3(X)$,
$\Phi_X = \pi_{\mathcal{P}X} \circ \psi_X : \mathcal{P}(X) \to \mathcal{P}^3(X)$,
then for any "neighbourhood" map $n : X \to \mathcal{P}^2(X)$ we get
$I_n = C(n) \circ \Omega_X : \mathcal{P}(X) \to \mathcal{P}(X)$ (interior),
$A_n = C(n) \circ \Phi_X : \mathcal{P}(X) \to \mathcal{P}(X)$ (adherence).
So ψ and π play two dual part in topology,
and
(Ω_X, Φ_X) is a kind of "universal topology" on X.

Dual monads : (Qual^-, Π^-) and (Qual^+, Π^+)

Qual^- : $f = (X, \mathcal{X}) \to (Y, \mathcal{Y})$, with $\mathcal{X} \subset \mathcal{P}(X)$, $\mathcal{Y} \subset \mathcal{P}(Y)$, $\mathcal{Y} \subset CC(f)\mathcal{X}$ (continuous map).

Qual^+ : $f = (X, \mathcal{X}) \to (Y, \mathcal{Y})$, with $\mathcal{X} \subset \mathcal{P}(X)$, $\mathcal{Y} \subset \mathcal{P}(Y)$, $\mathcal{P}C(f)\mathcal{Y} \subset \mathcal{X}$ (open map).

$\mathcal{X}^\cap = \mathcal{P}(\psi_X)(\mathcal{X}) \subset \mathcal{P}^2(X)$, $\mathcal{X}^\cup = C(\cup_X)(\mathcal{X}) \subset \mathcal{P}^2(X)$,
$\mathcal{X}^- = (\mathcal{X}^\cup)^\cap \subset \mathcal{P}^3(X)$, $\mathcal{X}^+ = (\mathcal{X}^\cap)^\cup \subset \mathcal{P}^3(X)$.

$\Pi^-(X, \mathcal{X}) = (CCX, \mathcal{X}^-)$, $\Pi^+(X, \mathcal{X}) = (CCX, \mathcal{X}^+)$.

$((\mathrm{Qual}^-)^{\Pi^-})^{\mathrm{op}} = \mathrm{Qual}^+$, $((\mathrm{Qual}^+)^{\Pi^+})^{\mathrm{op}} = \mathrm{Qual}^-$.

(by Damphousse-Guitart [1999] in the case of $\mathcal{E} = \mathrm{Ens}$).

Remark : These constructions “explain” the categorical meaning of the Vietoris distance and the Michael finite topology.

Regime of assimilations

On the necessity of possible causes : explanation of the causality axiom

— Let us assume that "Dominique assimilates Richard to John" is a given realized fact. Why is it so ? It could be because of the fact that Richard and John come from USA (and Dominique is sensible to this property). In this case, if Claude is sensible as Dominique to the property "to come from USA" — that is to say that from the point of view "to come from the USA" Claude is assimilated to Dominique — then Claude too assimilates Richard to John. So "to come from USA" is a possible cause of the given realized fact, but not necessarly the true one.

The causality axiom will express that a realized fact has at least one such a possible cause, or that *nothing is realized without a possible cause (i.e. the possibility of a cause)*. In the argumentation this principle is : nobody speaks without the possibility of a reason.

Definition of a 2-regime of assimilations

In any *algebraic universe* $\mathcal{E}$ — and a fortiori in any topos or any category of fuzzy sets — a 2-*regime of (binaries) assimilations* is a 2-dimensional data

$$R = \left[a_2 : E_2 \to \mathcal{P}(E_1^2); a_1 : E_1 \to \mathcal{P}(E_0^2\right],$$

where we assume a *causality axiom* :

$$\forall x, y \in E_0, \forall i \in E_1$$
$$(y, x) \in a_1(i) \Rightarrow$$
$$\exists c \in E_2 \forall j \in E_1 \left[(j, i) \in a_2(c) \Rightarrow (y, x) \in a_1(j)\right].$$

$(y, x) \in a_1(i)$? : from the point of view i, y is assimilated to x.

In english it means : if, from a point of view i, y is assimilated to x, then there is at least one possible cause c for that, i.e. a c such that if j is assimilated to i from this point of view c, then, from the point of view j, again y is assimilated to x.

Regimes as hyper-multimodal calculus

Given a binary relation $r \subset E \times F$ we get an adjoint pair

$$(-) \Uparrow r \dashv (-) \Downarrow r$$

with $(-) \Uparrow r : \mathcal{P}(E) \to \mathcal{P}(F)$:

$$X \mapsto X \Uparrow r = \{y \in F; \exists x(x \in X \,\&\, (y,x) \in r)\},$$

with $(-) \Downarrow r : \mathcal{P}(F) \to \mathcal{P}(E)$:

$$Y \mapsto Y \Downarrow r = \{x \in E; \forall y((y,x) \in r) \Rightarrow y \in Y)\}.$$

If $E = F$, $X \Downarrow r = \Box_r X$, $X \Uparrow r = \Diamond_{r^{\mathrm{op}}} X$, are necessity and co-possibility according to the point of view r.

With these notations, and with $[x', x] = \{i; (x', x) \in a_1(i)\}$, the axiom of causality could be expressed as

$$\forall x', x \in E_0 \left([x', x] \subset \cup_{c \in E_2} [x', x] \Downarrow a_2(c)\right).$$

The operators $\Uparrow$ and $\Downarrow$, ψ and π, $\exists$, $\forall$, ∇_r^+, ∇_r^-

In fact the basic operations for regimes, $\Uparrow$ and $\Downarrow$, are directly related to those of an algebraic universe :

$$(-) \Uparrow r = C(r^{\mathrm{op}}).\psi_E, \qquad (-) \Downarrow r = C(r).\pi_E,$$
$$(-) \Uparrow ((\alpha_F.f)^{\mathrm{op}}) \quad = C(f) = \quad (-) \Downarrow (\alpha_F.f),$$
$$\exists f = (-) \Uparrow (C(f).\alpha_F)^{\mathrm{op}}, \qquad \forall f = (-) \Downarrow (Cf.\alpha_F),$$
$$\exists f = C(Cf.\alpha_F).\psi_E, \qquad \forall f = C(Cf.\alpha_F).\pi_E.$$

Each relation of assimilation r determines two *gradients* :
$\nabla_r^+(X) = ((X \Uparrow r) \Downarrow r)\backslash X$,
$\nabla_r^-(X) = X\backslash((X \Downarrow r) \Uparrow r)$.
Slogan : The more an object E is structured, the more we get on it a strong regime of assimilations, and so a sophisticated *calculus of variations* through a complex system of gradients.

The natural regime on $\mathcal{P}(E)$, the convergence on $\mathcal{P}^2(E)$

If a set A is $A = \mathcal{P}(E)$, then A is structured by two basic binary relations, deduced from ψ_E and π_E, and a lot of derived relations.

So the Schmidt's Verzahnungoperator $(-)^*$ is given for $M \subset A$ by $M^* = \{X \subset E; \forall Z(Z \in M \Rightarrow Z \cap X \neq \emptyset)\} = \nu_{\mathcal{P}(E)}(M \Uparrow \pi_E)$.

The convergence of a filter $\mathcal{F}$ toward a filter $\mathcal{G}$ from the point of view of a topology $\mathcal{T}$ on E is described, with

$$\phi_E(\mathcal{F}) = C(\pi_E).\psi_{\mathcal{P}(E)}(\mathcal{F}),$$

by

$$\mathcal{T} \cap \mathcal{G} \subset \phi_E(\mathcal{F}).$$

Now if $\mathcal{F}, \mathcal{G}$ and $\mathcal{T}$ are arbitrary subsets of $\mathcal{P}(E)$ we get in this way the *regime of convergence* on $\mathcal{P}^2(E)$.

Regimes as presentations of spaces

Given a 1-*regime of (binaries) assimilations*

$$a : F \to \mathcal{P}(E^2)$$

we get a *qualification* $\mathcal{O}(a) = (E, \mathcal{O}[a])$, with $\mathcal{O}[a] \subset \mathcal{P}(E)$ the set of *a-open* subset U of E, i.e. subsets such that

$$U \subset \cup_{f\in F} U \Downarrow a(f).$$

In this way a 1-regime could be consider as a presentation of a space or of a qualification, as a quasi-uniform space is a presentation of a topological space.
The axiom of causality for a 2-regime means that $a_1^{-1} : E_0^2 \to \mathcal{P}(E_1)$ factorizes through $\mathcal{O}[a_2] \to \mathcal{P}(E_1)$. A kind of exactness condition.

$\mathcal{O}(I)$-sets or empirical sets

If I is a topological space, an $\mathcal{O}(I)$-set X could be seen as a regime, with $E_0 = X$, $E_1 = I$, $E_2 = \mathcal{O}(I)$, with $(x', x) \in a_1(i)$ iff i seen x' and x as confused, and with $(j, i) \in a_2(U)$ iff j and i belong to the open U of I.
In this case the causality axiom is true because of the fact that

$$[x', x] = \{j; (x', x) \in a_0(j)\}$$

is an open U of I; it means the *stability* of "confusing" points for observers $i \in I$.

Remark : in fact the general notion of regimes starts with the question : what is the part play by the axioms of a topology, and the part play by reflexivity and transitivity of equality in the construction of $\mathcal{O}(I)$-sets ? The natural answers are : the topology is nothing else than a second-level regime (with causality), and, in order to understand structures as regimes we need to assume <u>no</u> axioms at first for the assimilations.

Algebraic and geometric regimes

1 — If M is a monoïd, we get a regime with $E_2 = E_1 = E_0 = M$, $a_2 = a_1 = a$ with $(y, x) \in a(z)$ iff

$$y.z = x.$$

1 bis — An hypermonoïd $M \times M \to \mathcal{P}(M)$ determines a regime

$$M \to \mathcal{P}(M \times M).$$

2 — If P is an euclidian plane, we get a regime with $E_2 = E_1 = E_0 = P$, $a_2 = a_1 = a$ with $(y, x) \in a(z)$ iff

z, y, x are on a straight line, in this precise order.

2 bis — A plane curve Γ determines an assimilation γ on P given by

$(y, x) \in \gamma$ ifff the straight line (y, x) is tangent to Γ.

So, a family of plane curves determines a characteristic regime on the plane.

Regime of the metric plane, morphology, skeleton

The regime of the metric plane is the following one :
$E_0 = \mathbb{R}^2$, $E_1 = \mathbb{R}_{\geq 0}$, $E_2 = \{\omega\}$,
$(x', x) \in a_0(\epsilon)$ iff $d(x', x) < \epsilon$, and $(\epsilon', \epsilon) \in a_1(\omega)$ iff $\epsilon' > \epsilon$.
We have $\mathrm{int}X = \cup_{\epsilon>0} X \Downarrow \epsilon$, $\mathrm{adh}X = \cap_{\epsilon>0} X \Uparrow \epsilon$,
We define : $(x', x) \in a_1^{=}(\epsilon)$ iff $\forall q > \epsilon \left((x', x) \in a_1(q)\right)$.

The *skeleton* of an open X is (Motzkin 1935, Lantuejoul 1978) the locus of centers of disks in X which are tangent to $P \backslash X$ at at least two points.

It could be presented as :

$$Sq(X) = \cup_r \left(\cap_t \left[(X \Downarrow a_1(r) \backslash [[(X \Downarrow a_1(r)) \Downarrow a_1^{=}(t)] \Uparrow a_1^{=}(t)]] \right) \right).$$

NB : In this form, this definition makes sense for any regime.