Monoidal Quantaloids as a framework for (bi)categorical QM
Part I: Introduction

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A brief prehistory on the subject

- the notion of a monoidal quantaloid is *implicit* in the pioneering work of Carboni, Walters et al on *cartesian bicategories* (part I)
- Andrew Pitt in his "*Applications of Sup-lattice enriched category theory to Sheaf theory*" introduced *cartesian Suplattice-enriched categories* and its application in the *distributive categories of relations*
- from Quantales to Quantaloids (Mulvey and Rosenthal)
- only in Garraway's work on *Semi-quantaloids* **??**
- lot of activity on enrichement over commutative quantales (metric spaces etc.)
- re-ignited with Isar Stubbe's work, whose series of papers systematised quantaloidal calculus and enrichement

on the other hand the notion of monoidal bicategory is not new either (and quantaloids are particularly simple bicategories):

- monoidal *bicategories* eg. in Street's and Day's work, Gray monoids and so on, however the presentation is rather "awkward"
- perhaps the most "elementary" presentation of monoidal bicategories is quite recent: in Robin Houston's PhD thesis ("Linear Logic w/o units")

what is a quantaloid?

- A **Sup** enriched category
- a locally small complete and cocomplete partially ordered bicategory with colimits stable under composition in both sides
- a quantaloid is biclosed: both pre-composing and post-composing with an arrow $f: u \to v$ have right adjoints:

$$- \circ f \dashv \{f, -\} \quad \text{and} \quad f \circ - \dashv [f, -]$$

All these make a quantaloid an appropriate basis for *enrichement*

The quantaloid $\mathbf{Dist}(\mathcal{Q})$

has objects the categories *enriched* over $\mathcal{Q}$, namely:

- $\mathcal{Q}$-typed sets of objects with type: $ta = u \in \mathcal{Q}$
- homarrows $\mathsf{A}(a, a') \colon ta' \to ta$ satisfying:

  unit-inequalities: $\mathbf{1}_{ta} \leq \mathsf{A}(a, a)$

  transitivity: $\mathsf{A}(a, a') \circ \mathsf{A}(a', a'') \leq \mathsf{A}(a, a'')$

and arrows the *distributors* between $\mathcal{Q}$-categories, namely families of $\mathcal{Q}$-arrows satisfying:

$$\mathsf{B}(b', b) \circ \Phi(b, a) \leq \Phi(b', a)$$
$$\Phi(b, a) \circ \mathsf{A}(a, a') \leq \Phi(b, a')$$

$\mathbf{Dist}(\mathcal{Q})$ is the universal *direct-sum and split-monad* completion of $\mathcal{Q}$ in QUANT.

The locally preordered 2-category $\mathbf{Cat}(\mathcal{Q})$

has objects the categories enriched over $\mathcal{Q}$ and arrows *functors* between them, ie. an object mapping $\mathtt{F}\colon \mathcal{A}_0 \to \mathcal{B}_0$ satisfying:

type equalities $t(\mathtt{F}a) = ta,\ \forall a \in \mathsf{A}$

action inequalities $\mathsf{A}(a', a) \leq \mathsf{B}(\mathtt{F}a', \mathtt{F}a)$

$\mathbf{Dist}(\mathcal{Q})$ and $\mathbf{Cat}(\mathcal{Q})$ relate by means of this 2-functor:

$$\mathbf{Cat}(\mathcal{Q}) \longrightarrow \mathbf{Dist}(\mathcal{Q})\colon \mathtt{F} \longmapsto \mathsf{B}(-, \mathtt{F}-)$$

each functor $\mathtt{F}\colon \mathsf{A} \to \mathsf{B}$ induces a pair of adjoint distributors $\mathsf{B}(-, \mathtt{F}-) \dashv \mathsf{B}(\mathtt{F}-, -)$ which we denote as $\overrightarrow{\mathtt{F}}$, $\overleftarrow{\mathtt{F}}$ respectively

the advantage of being bicategorical!

is that the bicategorical notions and machinery are directly available in the quantaloidal calculus, namely things like:

- adjoints
- Kan extensions
- equivalent categories

and so on

The main definition

Definition (Monoidal quantaloid $(\mathcal{Q}, \otimes, i)$)

A *monoidal* quantaloid is a quantaloid $\mathcal{Q}$ endowed with a *tensor* product, namely a quantaloid homomorphism (ie. a **Sup**-enriched functor) $\otimes \colon \mathcal{Q} \times \mathcal{Q} \to \mathcal{Q}$ satisfying:

interchange law: for appropriate arrows $f : u \to v, g : v \to w, f' : u' \to v', g' : v' \to w'$ we get:

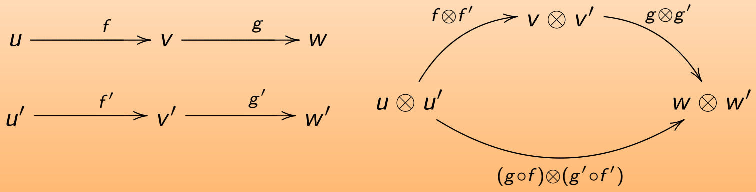
$$(g \otimes g') \circ (f \otimes f') = (g \circ f) \otimes (g' \circ f') \tag{1}$$

tensor preserves suprema:

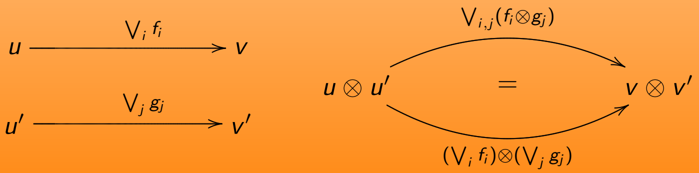
$$\bigvee_{i,j}(f_i \otimes g_j) = (\bigvee_i f_i) \otimes (\bigvee_j g_j) \tag{2}$$

associativity: isos $\alpha_{ABC} : (A \otimes B) \otimes C \cong A \otimes (B \otimes C)$

diagrammatically:



and:



a couple of remarks

the definition is a direct adaptation of the generic definition of a monoidal bicategory as a homomorphism $\mathcal{B} \times \mathcal{B} \to \mathcal{B}$ from which for instance the "interchange law" obtains immediately.
Also the *normality* of the tensor means that:

$$\mathbf{1}_{u \otimes v} = \mathbf{1}_u \otimes \mathbf{1}_v$$

for all objects u, v in $\mathcal{Q}$ which is really the most crucial ingredient in extending the monoidal structure to $\mathcal{Q}$-categories.
What enables the simplification of the generic definition is really the locally partially ordered structure of $\mathcal{Q}$: 2-cells are just inequalities, there are no non-identical isomorphic 1-cells and all diagrams involving 2-cells automatically commute.

Two very useful lemmas

Lemma (tensor "respects" order)

Given arrows $f, g\colon u \to v$ *with* $f \leq g$ *and* $h, k : u' \to v'$ *such that* $f \leq g$ *and* $h \leq k$ *we get that:*

$$f \otimes h \leq g \otimes h$$

A similar result holds for composition as well:

Lemma (composition respects order)

Given arrows $f, g\colon u \to v$ *with* $f \leq g$ *and* $k\colon w \to u$*,* $h\colon v \to z$ *such that*

$$f \circ k \leq g \circ k \ \ and \ \ h \circ f \leq h \circ g$$

then:

extend $\otimes$ to $\mathbf{Dist}(\mathcal{Q})$

this is really our main goal here, ie. to prove that:

Theorem (the monoidal quantaloid $(\mathbf{Dist}(\mathcal{Q}), \otimes, \mathsf{I})$)

If $\mathcal{Q}$ is a monoidal quantaloid, then $(\mathbf{Dist}(\mathcal{Q}), \otimes, \mathsf{I})$, with the tensor structure defined as in the Definition and I defined by the corresponding lemma is also monoidal

Defining the tensor for $\mathcal{Q}$-categories

Given $\mathcal{Q}$-categories A and B enriched over a monoidal quantaloid $(\mathcal{Q}, \otimes, i)$ we define:

the tensor A⊗B: to be the $\mathcal{Q}$-typed set having:

- **objects:** pairs (a, b) of objects $a \in \mathrm{A}_0$ and $b \in \mathrm{B}_0$ with type $t(a, b) = ta \otimes tb$
- **homarrows:** for objects $(a, b), (a', b')$ in A⊗B the homarrow $(\mathrm{A}{\otimes}\mathrm{B})((a, b), (a', b'))$ is given by the formula:

$$(\mathrm{A}{\otimes}\mathrm{B})((a, b), (a', b')) = \mathrm{A}(a, a') \otimes \mathrm{B}(b, b')$$

the tensor $\Phi \otimes \Psi$: of $\Phi\colon \mathrm{A} \longrightarrow \mathrm{C}$ and $\Psi\colon \mathrm{B} \longrightarrow \mathrm{D}$ between $\mathcal{Q}$-categories we define their tensor $\Phi \otimes \Psi\colon \mathrm{A}{\otimes}\mathrm{B} \to \mathrm{C}{\otimes}\mathrm{D}$ to be determined by the following family of $\mathcal{Q}$-arrows:

$$(\Phi{\otimes}\Psi)((c, d), (a, b)) = \Phi(c, a){\otimes}\Psi(d, b)\colon ta{\otimes}tb \to tc{\otimes}td$$

see the diagrams:

$$ta' \xrightarrow{A(a,a')} ta \qquad tb' \xrightarrow[B(b,b')]{} tb$$

$$ta' \otimes tb' \xrightarrow[:=A(a,a')\otimes B(b,b')]{(A\otimes B)((a,b),(a',b'))} ta \otimes tb$$

$$ta \xrightarrow{\Phi(c,a)} tc \qquad tb \xrightarrow[\Psi(d,b)]{} td$$

$$ta \otimes tb \xrightarrow[:=\Phi(c,a)\otimes\Psi(d,b)]{(\Phi\otimes\Psi)((c,d),(a,b))} tc \otimes td$$

Outline of the proof of the main theorem

In order to show that the structure defined in the aforegiven definition does endow **Dist**($\mathcal{Q}$) with a *tensor* someone should prove that:

- the $\mathcal{Q}$-typed set $A \otimes B$ *is* a $\mathcal{Q}$-category
- the tensor of distributors $\Phi \otimes \Psi$ *is* a distributor between the corresponding $\mathcal{Q}$-categories
- there exists a $\mathcal{Q}$-category I which is the **unit** for the tensor
- the tensor thus defined is indeed a quantaloidal *homomorphism* $\otimes$: **Dist**($\mathcal{Q}$) $\times$ **Dist**($\mathcal{Q}$) $\rightarrow$ **Dist**($\mathcal{Q}$) satisfying the "interchange" and the "preservation" properties
- the tensor is associative

The lemmas

This is indeed what we prove in a series of lemmas. The less trivial is perhaps to construct the isomorphisms satisfied by the unit:

Lemma (the unit of $(\mathbf{Dist}(\mathcal{Q}), \otimes)$:)

The unit for the tensor in $\mathbf{Dist}(\mathcal{Q})$ *is the* $\mathcal{Q}$*-category* I *having the single object* $*_i$ *of type* i *(*i *being the unit for the tensor in* $\mathcal{Q}$ *itself) and the single morphism* $\mathsf{I}(*_i, *_i) = \mathbf{1}_i = \mathbf{1}$*, is the unique (up to isomorphism) unit for the tensor in* $\mathbf{Dist}(\mathcal{Q})$ *with the (natural) isomorphisms* $\nu_A \colon \mathsf{A} \otimes \mathsf{I} \cong \mathsf{A}$ *and* $\mu_A \colon \mathsf{I} \otimes \mathsf{A} \cong \mathsf{A}$ *provided by appropriate distributors*

Tensor of presheaves

Lemma (tensor of presheaves)

Given presheaves $\phi\colon *_u \longrightarrow \mathsf{A}$ *and* $\psi\colon *_v \longrightarrow \mathsf{B}$ *their tensor is also a presheaf* $\phi \otimes \psi\colon \widehat{\mathbf{u} \otimes v} \longrightarrow \mathsf{A}{\otimes}\mathsf{B}$ *of type* $t(\phi \otimes \psi) = t\phi \otimes t\psi = u \otimes v$

Lemma (right lifting and extension of tensors)

Given presheaves $\phi\colon *_u \longrightarrow \mathsf{A}$, $\phi'\colon *_v \longrightarrow \mathsf{A}$ *and* $\psi\colon *_u \longrightarrow \mathsf{B}$ *and* $\psi'\colon *_v \longrightarrow \mathsf{B}$ *the right lifting* $[\phi \otimes \psi, \phi' \otimes \psi']$ *of their tensor satisfies the inequality:*

$$[\phi, \phi'] \otimes [\psi, \psi'] \leq [\phi \otimes \psi, \phi' \otimes \psi']$$

and given presheaves $\phi\colon *_u \longrightarrow \mathsf{A}$, $\phi'\colon *_u \longrightarrow \mathsf{C}$ *and* $\psi\colon *_v \longrightarrow \mathsf{B}$, $\psi'\colon *_v \longrightarrow \mathsf{C}$ *the right extension of the tensors satisfies:*

$$\{\phi, \phi'\} \otimes \{\psi, \psi'\} \leq \{\phi \otimes \psi, \phi' \otimes \psi'\}$$

the speciality of Cauchy presheaves

Lemma (tensor of Cauchy presheaves)

Given Cauchy presheaves $\phi\colon *_u \longrightarrow \mathsf{A}$ *and* $\psi\colon *_v \longrightarrow \mathsf{B}$ *on* A *and* B *respectively, their tensor* $\phi \otimes \psi$ *is also Cauchy, its adjoint given by* $\overline{\phi \otimes \psi} = \overline{\phi} \otimes \overline{\psi}$

which hold in general:

Theorem (tensor of adjoint distributors)

Given Cauchy distributors $\Phi\colon \mathsf{A} \longrightarrow \mathsf{C}$ *and* $\Phi\colon \mathsf{B} \longrightarrow \mathsf{D}$ *their tensor is also Cauchy, its adjoint be given by:*

$$\overline{\Phi \otimes \Psi} = \overline{\Phi} \otimes \overline{\Psi}$$

Theorem (lifting/extension of the tensor of Cauchy presheaves)

If the presheaves ϕ, ψ of the previous lemma are Cauchy, then the corresponding inequalities saturate to equalities, namely:

$$[\phi, \phi'] \otimes [\psi, \psi'] = [\phi \otimes \psi, \phi' \otimes \psi']$$
$$\{\phi, \phi'\} \otimes \{\psi, \psi'\} = \{\phi \otimes \psi, \phi' \otimes \psi'\}$$

which gives also as a direct consequence that for ϕ, ψ Cauchy presheaves

$$\|\phi \otimes \psi\| = \|\phi\| \otimes \|\psi\|$$

where by $\|\phi\|$ denotes the lifting $[\phi, \phi]$

why bicategories?

In the so called *dagger compact closed categories* model of Quantum Information/Computation someone works in a (symmetric) monoidal category and this explains why the "dagger" structure has to be imposed. There is not a notion of an adjoint arrow in a monoidal category! However a monoidal categrory is actually equivalent to (the suspension of) a single-object bicategory. Arrows in $\mathcal{V}$ are the 1-cells and the monoidal structure is the composition.
This motivates us to work directly with bicategories and with quantaloids in particular!

So recall that in the DCCCs models a *state* for instance is defined as an arrow $f : I \to A$ where I is the unit of the tensor. And that the *scalars* is precisely the (commutative in this case) category of the endoarrows $s : I \to I$ (since $\mathcal{V}$ is self-enriched!)
Is there an analogue in the monoidal quantaloid **Dist**($\mathcal{Q}$)? Of course! It is the presheaves ϕ: I $\to$ A.
Mind though that seen as a bicategory a monoidal category is single-object (or untyped) whereas a quanaloid is a many-object bicategory (so from this point of view the closest analogue to $\mathcal{V}$ would be a quantale!)
This motivates the following terminology:

- we call (all) presheaves over a $\mathcal{Q}$-category A *states* (those of type i are the *unital* ones)
- we call the category $\mathcal{P}$A the *free state space* over A

this "interpetation" means amongst other that:

- A quantum mechanical system A over the quantaloid $\mathcal{Q}$ is a $\mathcal{Q}$-enriched category, ie. an object of **Dist**($\mathcal{Q}$) (or **Cat**($\mathcal{Q}$) for that matter!)
- The Yoneda Lemma expresses the naturla embedding of any system A embedds naturally and *fully faithfully* into its free state space $\mathcal{P}$A

$$\mathbf{y}_{\mathsf{A}} : \mathsf{A} \hookrightarrow \mathcal{P}\mathsf{A}$$

- homarrows $\mathsf{A}(a, a')$ represent *transition amplitudes* for transitions $a \to a'$ in the system A.
- homarrows in the free state space $\mathcal{P}$A represent transition amplitude between states:

$$\mathcal{P}\mathsf{A}(\phi, \psi) = [\phi, \psi] = \|\phi \to \psi\|$$

Notice that for states of the *same type* these "amplitudes" *take values in the quantale* $\mathcal{Q}(u, u)$

- By the Yoneda Lemma, every presheaf is a colimit of representables:

$$\mathrm{colim}(\phi, \mathbf{y}_{\mathsf{A}}) = \mathbf{\Delta}\phi$$

  which motivates to interpet representables as "*eigenstates*" thus making the Yoneda Lemma sort of the categorical analogue of the Spectral Theorem. Colimits of representables are the analogue of the quantum superpositions.
- in fact we can *calculate* the components (projections) of a "state" ϕ along any eigenstate $\mathsf{A}(a, a) = \mathbf{e}_a$: they are simply: $\mathcal{P}\mathsf{A}(\mathsf{A}(a, -), \phi) = \phi(a)$
- the tensor category $\mathsf{A} \otimes \mathsf{B}$ is then the *composite* system and the corresponding state space is just $\mathcal{P}(\mathsf{A} \otimes \mathsf{B})$.

a setting to understand disentanglement

As an illustration of the plausibility of our approach let us present here some thoughts on *(dis)entanglement*!
Let us first ask: *what should we expect an entangled state over* A⊗B *to be* in $\mathbf{Dist}(\mathcal{Q})$?
Answer: It would be a state (ie. a presheaf) $\phi \colon *_u \to$ A⊗B which cannot be "analysed" into a tensor $\phi_a \otimes \phi_b$ of states over A and B!
Now let us try to "formalise" it a bit! To this end the following $\mathcal{Q}$-categories are naturally involved:

The categories $\mathcal{P}(\mathsf{A}\otimes\mathsf{B})$ and $\mathcal{P}\mathsf{A}\otimes\mathcal{P}\mathsf{B}$

The tensor of state spaces $\mathcal{P}\mathsf{A}\otimes\mathcal{P}\mathsf{B}$ has by definition:

objects : pairs $\phi_\mathsf{A}, \phi_\mathsf{B}$ of presheaves over the systems A and B respectively with type: $t(\phi_\mathsf{A}, \phi_\mathsf{B}) = t\phi_\mathsf{A} \otimes t\phi_\mathsf{B}$

homarrows: for pairs of presheaves $(\phi_\mathsf{A}, \phi_\mathsf{B})$ and $(\psi_\mathsf{A}, \psi_\mathsf{B})$ we have:

$$\begin{aligned}(\mathcal{P}\mathsf{A}\otimes\mathcal{P}\mathsf{B})((\phi_\mathsf{A}, \phi_\mathsf{B}), (\psi_\mathsf{A}, \psi_\mathsf{B})) &= \\ \mathcal{P}\mathsf{A}(\phi_\mathsf{A}, \psi_\mathsf{A}) \otimes \mathcal{P}\mathsf{B}(\phi_\mathsf{B}, \psi_\mathsf{B}) &= \\ [\phi_\mathsf{A}, \psi_\mathsf{A}] \otimes [\phi_\mathsf{B}, \psi_\mathsf{B}]\end{aligned}$$

On the other hand consider $\mathcal{P}(\mathsf{A}\otimes\mathsf{B})$, ie. the category of (contravariant) presheaves over the tensor category $\mathsf{A}\otimes\mathsf{B}$ having:

objects: presheaves $\phi\colon *_u \longrightarrow \mathsf{A}\otimes\mathsf{B}$

homarrows: $(\mathcal{P}(\mathsf{A}\otimes\mathsf{B}))(\phi, \psi) = [\phi, \psi]$

the (Dis)entanglement functor

In order to inquire how these two $\mathcal{Q}$-categories relate, we introduce the assignment $\mathtt{D}\colon \mathcal{P}\mathsf{A}\otimes\mathcal{P}\mathsf{B} \longrightarrow \mathcal{P}(\mathsf{A}\otimes\mathsf{B})$ which acts as follows:

$$\mathtt{D}\colon (\phi_{\mathsf{A}}, \phi_{\mathsf{B}}) \mapsto \phi_{\mathsf{A}} \otimes \phi_{\mathsf{B}}$$

ie. mapping the object $(\phi_{\mathsf{A}}, \phi_{\mathsf{B}})$, the pair of presheaves, to their *tensor* $\phi_{\mathsf{A}} \otimes \phi_{\mathsf{B}}$ which is indeed a state over $\mathsf{A}\otimes\mathsf{B}$) In the first place we prove:

Theorem

The assignment defined $\mathtt{D}\colon \mathcal{P}\mathsf{A}\otimes\mathcal{P}\mathsf{B} \longrightarrow \mathcal{P}(\mathsf{A}\otimes\mathsf{B})$ *above is functorial. We call it the* **disentanglement** *functor.*

but there is more!

The categories $\mathcal{P}A \otimes \mathcal{P}B$ and $\mathcal{P}(A \otimes B)$ "engage" naturally the functors:

$$\mathbf{y}_{A\otimes B} : A \otimes B \to \mathcal{P}(A \otimes B)$$
$$\mathbf{y}_A \otimes \mathbf{y}_B : A \otimes B \to \mathcal{P}A \otimes \mathcal{P}B$$

Lemma (The left Kan extension $\langle \mathbf{y}_{A\otimes B}, \mathbf{y}_A \otimes \mathbf{y}_B \rangle$:)

The left Kan extension $\langle \mathbf{y}_{A\otimes B}, \mathbf{y}_A \otimes \mathbf{y}_B \rangle$ given by the colimit $\mathrm{colim}(\overleftarrow{\mathbf{y}_A \otimes \mathbf{y}_B}, \mathbf{y}_{A\otimes B})$ of the "tensor" Yoneda embedding $\mathbf{y}_{A\otimes B}$ weighted by the right adjoint distributor induced by $\mathbf{y}_A \otimes \mathbf{y}_B$, always exists and it is just the (dis)entanglement functor itself:

$$\mathrm{colim}(\overleftarrow{\mathbf{y}_A \otimes \mathbf{y}_B}, \mathbf{y}_{A\otimes B}) \cong \mathtt{D}$$

the left Kan extension $\langle \mathbf{y}_A \otimes \mathbf{y}_B, \mathbf{y}_{A\otimes B} \rangle = \mathbf{y}_\otimes$

However the left Kan extenstion $\langle \mathbf{y}_A \otimes \mathbf{y}_B, \mathbf{y}_{A\otimes B} \rangle$ does **not** always exist! In fact its existence imposes a "disentanglement" condition: $\mathrm{colim}(\phi, \mathbf{y}_A \otimes \mathbf{y}_B)$ exists iff there exist a (constant) functor $\Delta_{(\phi_A,\phi_B)} : *_{t\phi} \longrightarrow \mathcal{P}A \otimes \mathcal{P}B$ picking out the object (ϕ_A, ϕ_B) of $A \otimes B$, necessarily of type: $t(\phi_A, \phi_B) = t\phi_A \otimes t\phi_B = t\phi$ such that:

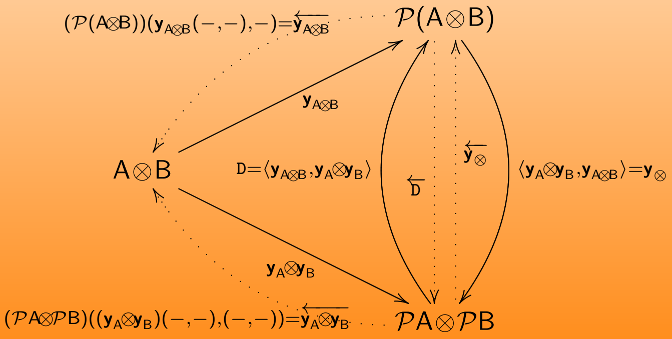
$$\forall \psi_A, \psi_B \in \mathcal{P}(A \otimes B) : \overleftarrow{\mathbf{\Delta}_{(\phi_A,\phi_B)}} = [\phi, \overleftarrow{\mathbf{y}_A \otimes \mathbf{y}_B}]$$

which after some calculations proves to be equivalent to:

$$[\phi_A, \psi_A] \otimes [\phi_B, \psi_B] = [\phi, \psi_A \otimes \psi_B]$$

the disentanglement adjunction: $\mathbf{y}_{\otimes} \dashv \mathtt{D}$

Therefore if the "disentanglement" condition $[\phi_A, \psi_A] \otimes [\phi_B, \psi_B] = [\phi, \psi_A \otimes \psi_B]$ is satisfied **for all** ϕ then $\mathbf{y}_{\otimes} \dashv \mathtt{D}$:



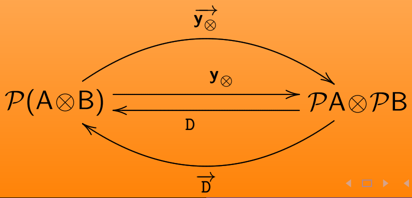
and we get the *factorization*: $\mathbf{y}_{A\otimes B} \cong \mathtt{D} \circ (\mathbf{y}_A \otimes \mathbf{y}_B)$

Disentanglement as the equivalence $\mathcal{P}(\mathsf{A}\otimes\mathsf{B}) \simeq \mathcal{P}\mathsf{A}\otimes\mathcal{P}\mathsf{B}$

However we want disentanglement to mean: for *every* state ϕ on $\mathsf{A}\otimes\mathsf{B}$ there is a (unique) pair of presheaves $(\phi_{\mathsf{A}}, \phi_{\mathsf{B}})$ on A and B respectively which *is precisely* the $\mathrm{colim}(\phi, \mathbf{y}_{\mathsf{A}} \otimes \mathbf{y}_{\mathsf{B}})$ such that $\phi = \phi_{\mathsf{A}} \otimes \phi_{\mathsf{B}}$. then we must require that the adjunction $\mathbf{y}_{\otimes} \dashv \mathtt{D}$ is in fact an equivalence! So we define:

Definition

The system $\mathsf{A}\otimes\mathsf{B}$ is *disentangled* (or *decomposable* or *separable*) if the following diagram is an equivalence in $\mathbf{Cat}(\mathcal{Q})$:



Prospectus

- how far the analogy with DCCCs can go? This seems to relate to the connection with *involutive* quanaloids as well
- applications to quantales (quantales are single-object quantaloids
- applications to $\mathcal{Q}$-orders (or $\mathcal{Q}$-sheaves) ($\mathbf{Ord}(\mathcal{Q})$ is basically $\mathbf{Map}(\mathbf{Dist}(\mathcal{Q}_{\mathsf{si}}))$ and $\mathcal{Q}_{\mathsf{si}}$ hence $\mathbf{Dist}(\mathcal{Q}_{\mathsf{si}})$ are also monoidal)
- and of course how do Cartesian quantaloids fit in this framework! For instance: how does our "disentanglement" condition in terms of the equivalence $\mathcal{P}(\mathsf{A} \otimes \mathsf{B}) \simeq \mathcal{P}\mathsf{A} \otimes \mathcal{P}\mathsf{B}$ relate to the Frobenius-separability condition in the context of cartesian bicategories?

Some references:

Only indicative!

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