A double categorical analysis of the tripos-to-topos construction

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Outline

- Universal characterization of the tripos-to-topos construction
- Decomposition of the tripos-to-topos construction
- Double categorical interpretation of the formalism

The tripos to topos construction

- The tripos-to-topos construction was defined in 1980 by Hyland, Johnstone and Pitts [4] as a tool to construct the effective topos.
- It allows to construct interesting toposes that are not Grothendieck toposes.
- It relates two classes of models of intuitionistic higher order logic (triposes and toposes).

Definition (Heyting Algebra)

A **Heyting Algebra** is a poset that is bicartesian closed as a category. The category **HA** of Heyting algebras has monotone maps that preserve all structure ($\top, \wedge, \bot, \vee, \Rightarrow$) as morphisms.

Definition (Tripos)

Let $\mathbb{C}$ be a category with finite limits. A **tripos over** $\mathbb{C}$ is a functor

$$\mathcal{P} : \mathbb{C}^{\text{op}} \to \textbf{HA},$$

such that

1. For all $f : A \to B$ in $\mathbb{C}$, the maps $\mathcal{P}(f) : \mathcal{P}(B) \to \mathcal{P}(A)$ have left and right adjoints
$$\exists_f \dashv \mathcal{P}(f) \dashv \forall_f$$
subject to the **Beck-Chevalley condition.**
2. For each $A \in \mathbb{C}$ there exists $\pi(A) \in \mathbb{C}$ and $(\ni_A) \in \mathcal{P}(\pi(A) \times A)$ such that for all $\psi \in \mathcal{P}(C \times A)$ there exists $\chi_\psi : C \to \pi(A)$ such that
$$\mathcal{P}(\chi_\psi \times A)(\ni_A) = \psi.$$

Examples of triposes

- The **Kleene realizability tripos** is given by the functor

$$\mathbf{eff} \;=\; \mathbf{Set}(-,\mathbf{P}(\mathbb{N})) \;:\; \mathbf{Set}^{\mathrm{op}} \to \mathbf{HA}$$

where for $\varphi, \psi \in \mathbf{Set}(I, \mathbf{P}(\mathbb{N}))$ the order relation is defined by

$$\varphi \leq \psi \quad :\Leftrightarrow \quad \exists f \text{ primitive recursive } \forall i : I\ \forall n \in \varphi(i)\,.\, f(n) \in \psi(i).$$

(Strictly speaking, this gives a preorder, so we have to quotient out the symmetric part.)

- For a **complete Heyting algebra** A, the functor

$$\mathcal{P}_A \;=\; \mathbf{Set}(-, A) \;:\; \mathbf{Set}^{\mathrm{op}} \to \mathbf{HA}$$

is a tripos if we equip the sets $\mathbf{Set}(I, A)$ with the pointwise ordering.

A topos from a tripos

For a tripos $\mathcal{P}$ on $\mathbb{C}$, we can construct a topos $\boldsymbol{T}\mathcal{P}$ as follows:

- The **objects** of $\boldsymbol{T}\mathcal{P}$ are pairs $(A, \sim)$, where $A \in \text{obj}(\mathbb{C})$, $(\sim) \ \in \mathcal{P}(A \times A)$, and the judgments

$$x \sim y \vdash y \sim x$$
$$x \sim y, y \sim_A z \vdash x \sim z$$

  hold in the logic of $\mathcal{P}$.

  Intuition: "$\sim$ is a partial equivalence relation on A in the logic of $\mathcal{P}$"

A topos from a tripos

- A **morphism** from $(A, \sim)$ to $(B, \sim)$ is a predicate $\phi \in \mathcal{P}(A \times B)$ such that the following judgments hold in $\mathcal{P}$.

$$
\begin{array}{lr}
\text{(strict)} & \phi(x, y) \vdash x \sim x \wedge y \sim y \\
\text{(cong)} & \phi(x, y), x \sim x', y \sim y' \vdash \phi(x', y') \\
\text{(singval)} & \phi(x, y), \phi(x, y') \vdash y \sim y' \\
\text{(tot)} & x \sim x \vdash \exists y \,.\, \phi(x, y)
\end{array}
$$

A topos from a tripos

- The **composition** of two morphisms

$$(A, \sim) \xrightarrow{\phi} (B, \sim) \xrightarrow{\gamma} (C, \sim),$$

  is given by

$$(\gamma \circ \phi)(a, c) \quad \equiv \quad \exists b \,.\, \phi(a, b) \wedge \gamma(b, c).$$

- The **identity** morphism on $(A, \sim)$ is $\sim$.

Examples

- For the Kleene realizability tripos **eff**, the topos $\boldsymbol{T}(\mathbf{eff})$ is Hyland's **effective topos**.
- For a comlete Heyting algebra A, we have

$$\boldsymbol{T}\mathcal{P}_A \simeq \mathbf{Sh}(A),$$

where $\mathbf{Sh}(A)$ is the topos of sheaves on A.

- We want a universal characterization of this construction.
- This should take place in a 2-dimensional framework.
- We will now introduce the relevant 2-categories of triposes and toposes.

Tripos morphisms

- A **tripos morphism** between triposes $\mathcal{P} : \mathbb{C}^{op} \to \mathbf{HA}$ and $\mathcal{Q} : \mathbb{D}^{op} \to \mathbf{HA}$ is a pair (F, Φ) of a functor

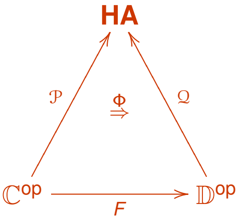
$$F : \mathbb{C} \to \mathbb{D}$$

and a natural transformation

$$\Phi : \mathcal{P} \to \mathcal{Q} \circ F$$

such that

1. F preserves finite products
2. For every $C \in \mathbb{C}$, Φ_C preserves finite meets.



- If Φ commutes with existential quantification, i.e.

$$\Phi_D(\exists_f \psi) = \exists_{Ff} \Phi_C(\psi)$$

for all $f : C \to D$ in $\mathbb{C}$ and $\psi \in \mathcal{P}(C)$, then we call the tripos morphism **regular**.

2-cells of triposes

A 2-cell

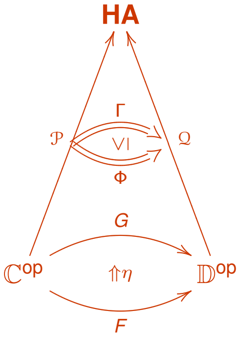
$$\eta : (F, \Phi) \to (G, \Gamma) : \mathcal{P} \to \mathcal{Q}$$

is a natural transformation

$$\eta : F \to G$$

such that for all $C \in \mathbb{C}$ and all $\psi \in \mathcal{P}(C)$, we have

$$\Phi_C(\psi) \ \leq \ \mathcal{Q}(\eta_C)(\Gamma_C(\psi)).$$



The 2-categories **Trip** and $\mathbf{Trip}_r$ of triposes

- The 2-category **Trip** consists of triposes, tripos morphisms and tripos transformations.
- The 2-category $\mathbf{Trip}_r$ consists of triposes, **regular** tripos morphisms, and tripos transformations.
- There is an inclusion

$$\mathbf{Trip}_r \hookrightarrow \mathbf{Trip}$$

which is locally fully faithful and identity on objects.

The 2-categories **Top** and **Top**$_r$ of toposes

- The 2-category **Top** of toposes consists of
  - toposes,
  - *finite limit preserving* functors, and
  - arbitrary natural transformations
- The 2-category **Top**$_r$ is consists of
  - toposes,
  - *regular (i.e., finite limit and epi preserving)* functors, and
  - arbitrary natural transformations.
- There is an inclusion

  $$\mathbf{Top}_r \hookrightarrow \mathbf{Top}$$

  which is locally fully faithful and identity on objects.

The functor $S : \mathbf{Top} \to \mathbf{Trip}$

- For a given topos $\mathcal{E}$, the functor $\mathcal{E}(-, \Omega)$ is a tripos if we equip the homsets with the inclusion ordering of the classified subobjects
- This construction is 2-functorial and gives rise to a 2-functor

$$S : \mathbf{Top} \to \mathbf{Trip}$$

which factors through the inclusions of $\mathbf{Top}_r$ and $\mathbf{Trip}_r$:

$$\begin{array}{ccc} \mathbf{Top}_r & \xrightarrow{S} & \mathbf{Trip}_r \\ \big\downarrow & & \big\downarrow \\ \mathbf{Top} & \xrightarrow[S]{} & \mathbf{Trip} \end{array}$$

The functor $\boldsymbol{S}$: **Top** $\to$ **Trip**

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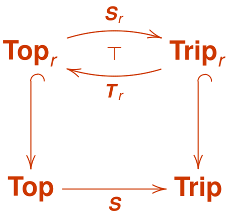
- Obvious question: Is the tripos-to-topos construction left (bi)adjoint to $\boldsymbol{S}$?

2 answers:

- $\mathbf{S} : \mathbf{Top}_r \to \mathbf{Trip}_r$ has a left biadjoint $\mathbf{T} : \mathbf{Trip}_r \to \mathbf{Top}_r$ whose object part is the tripos-to-topos construction.



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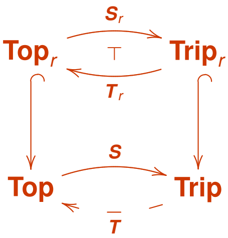

- However, already in the 1980 paper [4] of Hyland, Johnstone and Pitts, the construction is described for arbitrary tripos morphisms.
- In the general case, we obtain an **oplax** functor $T : \mathbf{Trip} \to \mathbf{Top}$.
- This extension is important in categorical realizability, as it allows to construct topologies and geometric morphisms on toposes from topologies and geometric morphisms on triposes.
- We show an example of how oplaxness occurs.

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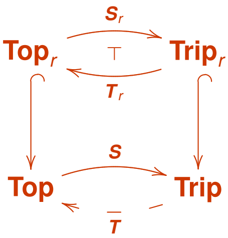

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- We show an example of how oplaxness occurs.

Example

- $\mathbb{B}$ is the 2-element Heyting algebra $\mathbb{B} = \{\text{true}, \text{false}\}$ with $\text{false} \leq \text{true}$.
- $$\mathbb{B} \xrightarrow{\quad\delta\quad} \mathbb{B} \times \mathbb{B} \xrightarrow{\quad\wedge\quad} \mathbb{B}$$

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$$\mathbf{Sh}(\mathbb{B}) \sim \mathbf{Set} \qquad \mathbf{Sh}(\mathbb{B} \times \mathbb{B}) \sim \mathbf{Set} \times \mathbf{Set} \qquad \mathbf{Sh}(\mathbb{B}) \sim \mathbf{Set}$$

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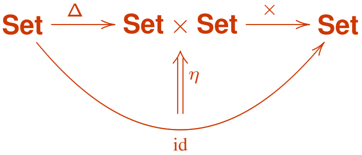
$$\mathbf{Sh}(\mathbb{B}) \sim \mathbf{Set} \xrightarrow[\Delta]{} \mathbf{Sh}(\mathbb{B} \times \mathbb{B}) \sim \mathbf{Set} \times \mathbf{Set} \xrightarrow[\times]{} \mathbf{Sh}(\mathbb{B}) \sim \mathbf{Set}$$

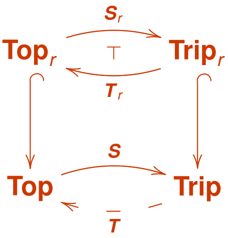
Example

- Comparing the composition of the images of the tripos transformations with the image of the composition we get



- This shows that the tripos-to-topos construction is only oplax functorial.

The functor T



- Since T is oplax, it can't be biadjoint to S in the ordinary sense.
- However, T and S form **generalized biadjunction**, in a sense that we will explain now.

Pre-equipments

Definition (Pre-equipment)

- A **pre-equipment** is given by a 2-category $\mathscr{C}$ together with a designated subclass $\mathscr{C}_r$ of the class of all 1-cells which contains identities and is closed under composition and vertical isomorphisms.
- Elements of $\mathscr{C}_r$ are called **regular 1-cells**.
- We call a pre-equipment **geometric**, if all left adjoints in it are regular.

- Pre-equipments generalize proarrow equipments, introduced by Woods in 1982 [7].

Pre-equipments

Definition (Morphism of pre-equipments)

A **morphism of pre-equipments** $\mathscr{C}$ and $\mathscr{D}$ is an oplax functor $F : \mathscr{C} \to \mathscr{D}$ such that

- Ff is a regular 1-cell whenever f is a regular 1-cell
- all identity constraints $FI_A \to I_{FA}$ are invertible, and
- composition constraints $F(gf) \to Fg\,Ff$ are invertible whenever g is a regular 1-cell.
- We call a morphism of pre-equipments **strong**, if all constraint cells are invertible.

- A special case of pre-equipment morphisms has been introduced in ‘Dualizations and antipodes’ [2] by Day, McCrudden and Street under the name **special lax functor** in the context of categories of profunctors.
- In the context of double categories, similar notions have been studied by Verity, Paré, Pronk, Shulman, and others (more on this later).

Pre-equipments

Definition (Transformation of pre-equipments)

A **transformation of pre-equipments** between pre-equipment morphisms F, G is an oplax natural transformation $\eta : F \to G$ such that

- all η_A are regular, and
- $\eta_B\, Ff \to Gf\, \eta_A$ is invertible whenever $f : A \to B$ is regular.

$$A \longrightarrow B \qquad\qquad A \xrightarrow{\text{reg}} B$$

$$\begin{array}{ccc} FA & \to & FB \\ {\scriptstyle \text{reg}}\downarrow & \Downarrow & \downarrow{\scriptstyle \text{reg}} \\ GA & \to & GB \end{array} \qquad\qquad \begin{array}{ccc} FA & \xrightarrow{\text{reg}} & FB \\ {\scriptstyle \text{reg}}\downarrow & \cong & \downarrow{\scriptstyle \text{reg}} \\ GA & \xrightarrow[\text{reg}]{} & GB \end{array}$$

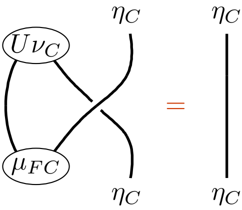
- Transformations of *strong* pre-equipment morphisms have been studied by Johnstone in *'Fibrations and partial products in a 2-category'* [5] in the context of lax slice categories (This has inspired the present work).
- Again, very similar notions for double categories

Biadjunctions of pre-equipments

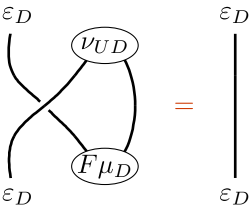
A **biadjunction** between pre-equipments $\mathscr{C}$ and $\mathscr{D}$ is given by

- pre-equipment morphisms $F : \mathscr{C} \to \mathscr{D}$ $\quad U : \mathscr{D} \to \mathscr{C}$,
- pre-equipment transformations $\eta : \mathrm{id}_{\mathscr{C}} \to UF$ $\quad \varepsilon : FU \to \mathrm{id}_{\mathscr{D}}$
- **invertible** modifications $\mu : \mathrm{id}_U \to U\varepsilon \circ \eta U$ $\quad \nu : \varepsilon F \circ F\eta \to \mathrm{id}_F$

such that the equalities



and



hold for all $C \in \mathscr{C}$ and $D \in \mathscr{D}$.

Properties of biadjunctions of pre-equipments

- If they exist, biadjoints are unique up to equivalence.
- For any biadjunction $F \dashv U$, the right adjoint U is **strong**.

Main result

- **Top**, with regular functors as regular 1-cells is a geometric pre-equipment.
- **Trip**, with regular tripos morphisms as regular 1-cells is a geometric pre-equipment.
- $\boldsymbol{S} : \mathbf{Top} \to \mathbf{Trip}$ is a strong morphism of equipments.

Theorem

The tripos-to-topos construction gives rise to an equipment morphism $\boldsymbol{T} : \mathbf{Trip} \to \mathbf{Top}$ which is left biadjoint to $\boldsymbol{S}$.

$$\boldsymbol{T} \dashv \boldsymbol{S} : \mathbf{Top} \to \mathbf{Trip}$$

- In the construction of $\boldsymbol{T}(F, \Phi)$ for general (F, Φ), we have to deal with ‘weakly complete objects’, which make things complicated
- Attempts to give an easier proof naturally lead to a decomposition of the construction.

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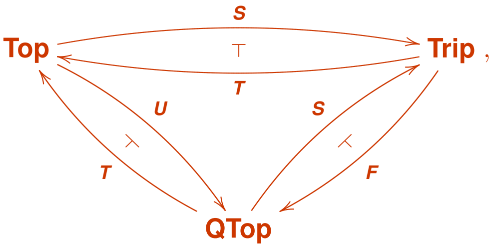
- 

- **QTop** is the pre-equipment of **q-toposes**.
- Q-toposes are a weakened version of quasi-toposes.

Q-Toposes

Definition

- A monomorphism $e : U \to B$ in a category $\mathcal{C}$ is called **strong**, if for every commutative square $\begin{array}{ccc} A & \to & U \\ {\scriptstyle e}\downarrow & \overset{h}{\nearrow} & \downarrow{\scriptstyle m} \\ Q & \to & B \end{array}$ where e is an epimorphism, there exists a (unique) h.

Definition

- A **q-topos** is a category $\mathcal{C}$ with finite limits, an exponentiable classifier of strong monomorphisms, and pullback stable quotients of strong equivalence relations.
- The **pre-equipment QTop** consists of
  - q-toposes as **objects**,
  - finite limit preserving functors as **1-cells**,
  - 1-cells that preserve epimorphisms and regular epimorphisms as **regular 1-cells**, and
  - arbitrary natural transformations as **2-cells**.

Q-Toposes

The q-topos induced by a tripos

Given a tripos $\mathcal{P} : \mathbb{C}^{op} \to \mathbf{HA}$, we define a category $\boldsymbol{F}\mathcal{P}$

- Objects are the same as in $\boldsymbol{T}\mathcal{P}$, i.e., pairs $(C, \sim)$ with $\sim \in \mathcal{P}(C \times C)$ such that $x \sim y, y \sim z \vdash x \sim z$ and $x \sim y \vdash y \sim x$.
- Morphisms of type $(C, \sim) \to (D, \sim)$ are morphisms $f : C \to D$ in $\mathbb{C}$ such that

$$x \sim y \vdash fx \sim fy,$$

  quotiented by an equivalence relation:
  f, g are identified, iff

$$x \sim x \vdash fx \sim gx$$

- Composition and identities are inherited from $\mathbb{C}$.

Lemma

For a tripos $\mathcal{P}$, $\boldsymbol{F}\mathcal{P}$ is a q-topos.

Q-Toposes

Functors from tripos morphisms

- Given a tripos-morphism

$$(F, \Phi) : \mathcal{P} \to \mathcal{Q},$$

  we define the functor

$$\boldsymbol{F}(F, \Phi) : \boldsymbol{F}\mathcal{P} \to \boldsymbol{F}\mathcal{Q}$$

  by

$$\begin{array}{ccc} (C, \sim) & \mapsto & (FC, \Phi \sim) \\ (f : (C, \sim) \to (D, \sim)) & \mapsto & Ff \end{array}$$

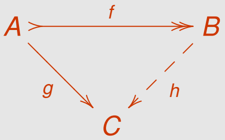
- These constructions give rise to a **2-functor** $\boldsymbol{F} : \mathbf{Trip} \to \mathbf{QTop}$.

Q-Toposes

Coarse objects

Definition

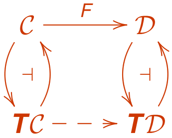
$C \in \mathcal{C}$ is called **coarse**, if for every $f : A \rightarrowtail\!\!\twoheadrightarrow B$ which is monic and epic, and for all $g : A \to C$, there exists $h : B \to C$ such that $hf = g$.



- The coarse objects in a q-topos form a reflective subcategory with cartesian rector, which is a topos! (Well known for quasitoposes)

The functor T : QTop $\to$ Top

- Given a functor between q-toposes, we may compose it with the appropriate components of the reflections to obtain a functor between the subtoposes of coarse objects.



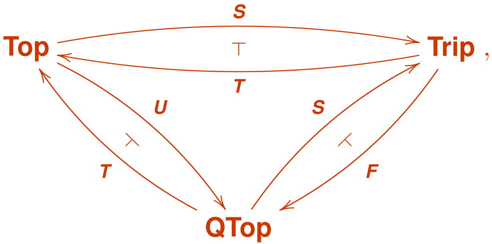
- This operation gives rise to a pre-equipment morphism

$$T : \textbf{QTop} \to \textbf{Top}$$

which is left biadjoint to the canonical embedding

$$U : \textbf{Top} \to \textbf{QTop}$$

- Coming back to,



we still have to explain how to construct S : **QTop** $\to$ **Trip**

- The important observation for that is that — a bit surprisingly — the fibration of strong monomorphisms of a q-topos is a tripos. This can be proven by defining an internal language in the style of Lambek-Scott.

Types:

$$A ::= X \mid 1 \mid \Omega \mid PA \mid A \times A \qquad X \in \mathrm{obj}(\mathcal{C})$$

Terms:
We use Δ to denote a context $x_1{:}A_1, \ldots, x_n{:}A_n$ of typed variables.

$$\frac{}{\Delta \mid x_i : A_i}\ (i{=}1,\ldots,n) \qquad \frac{}{\Delta \mid * : 1}$$

$$\frac{\Delta, x{:}A \mid \varphi[x] : \Omega}{\Delta \mid \{x|\varphi[x]\} : PA} \qquad \frac{\Delta \mid a : A \qquad \Delta \mid b : B}{\Delta \mid (a,b) : A \times B}$$

$$\frac{\Delta \vdash a : A \qquad \Delta \vdash M : PA}{\Delta \vdash a \in M : \Omega} \qquad \frac{\Delta \vdash a : A \qquad \Delta \vdash a' : A}{\Delta \vdash a = a' : \Omega}$$

$$\frac{\Delta \mid a : X}{\Delta \mid f(a) : Y}\ f \in \mathcal{C}(X, Y)$$

Deduction rules:

$$\frac{}{\Delta \mid p_1, \ldots, p_n \vdash p_i}\ \text{Ax}\ (i{=}1,\ldots,n) \qquad \frac{\Delta \mid \Gamma \vdash p \qquad \Delta \mid \Gamma, p \vdash q}{\Delta \mid \Gamma \vdash q}\ \text{Cut}$$

$$\frac{}{\Delta \mid \Gamma \vdash t = t}\ {=}\text{R} \qquad \frac{\Delta, x{:}A \mid \Gamma \vdash \varphi[x,x]}{\Delta \mid \Gamma, s = t \vdash \varphi[s,t]}\ {=}\text{L}$$

$$\frac{\Delta, x{:}A \mid \Gamma \vdash p[x] = (x \in M)}{\Delta \mid \Gamma \vdash \{x|p[x]\} = M}\ \text{P-}\eta \qquad \frac{}{\Delta \mid \Gamma \vdash (a \in \{x|p[x]\}) = p[a]}\ \text{P-}\beta$$

$$\frac{}{\Delta \mid \Gamma \vdash t = *}\ 1\text{-}\eta \qquad \frac{\Delta \mid \Gamma, p \vdash q \qquad \Delta \mid \Gamma, q \vdash p}{\Delta \mid \Gamma \vdash p = q}\ \text{Ext}$$

Analyzing the unit of $\boldsymbol{T} \dashv \boldsymbol{S}$

The unit of $\boldsymbol{T} \dashv \boldsymbol{S} : \mathbf{Top} \to \mathbf{Trip}$ gives rise to 1-cells $(D, \Delta) : \mathcal{P} \to \boldsymbol{ST}\mathcal{P}$

and to 2-cells
$$\begin{array}{ccc} \mathcal{P} & \xrightarrow{(F,\Phi)} & \mathcal{R} \\ \downarrow & \Downarrow & \downarrow \\ \boldsymbol{ST}\mathcal{P} & \longrightarrow & \boldsymbol{ST}\mathcal{R} \end{array}$$
which decompose into
$$\begin{array}{ccc} \mathcal{P} & \xrightarrow{(F,\Phi)} & \mathcal{R} \\ \downarrow & \Downarrow\alpha & \downarrow \\ \boldsymbol{SF}\mathcal{P} & \longrightarrow & \boldsymbol{SF}\mathcal{R} \\ \downarrow & \Downarrow\beta & \downarrow \\ \boldsymbol{ST}\mathcal{P} & \longrightarrow & \boldsymbol{ST}\mathcal{R} \end{array} .$$

Lemma

α is an isomorphism whenever Φ commutes with $\exists$ along diagonal mappings $\delta : A \to A \times A$, and β is an isomorphism whenever Φ commutes with $\exists$ along projections. Furthermore, α is always an epimorphism and β is always a monomorphism.

Example

The tripos transformation $\mathcal{P}_{\wedge} : \mathcal{P}_{\mathbb{B}\times\mathbb{B}} \to \mathcal{P}_{\mathbb{B}}$ commutes with $\exists$ along δ. Therefore we have

$$
\begin{array}{ccc}
\mathbf{Set} & \xrightarrow{\ \mathrm{id}\ } & \mathbf{Set} \\
\downarrow & \cong & \downarrow \\
\mathbf{Sep}(\mathbb{B}\times\mathbb{B}) & \longrightarrow & \mathbf{Sep}(\mathbb{B}) \\
\downarrow & \Downarrow\beta & \downarrow \\
\mathbf{Set}\times\mathbf{Set} & \longrightarrow & \mathbf{Set}
\end{array}
\;.
$$

Example: Modified realizability

The embedding

$$\nabla = (\neg\neg \circ \Delta) \quad : \quad \mathcal{P}_{\mathbb{B}} \to \mathbf{mr}$$

of the classical predicates into the modified realizability tripos **mr** commutes with $\exists$ along projections. This gives

$$\begin{array}{ccc} \mathbf{Set} & \xrightarrow{\mathrm{id}} & \mathbf{Set} \\ \downarrow & \Downarrow\alpha & \downarrow \\ \boldsymbol{F}(\mathcal{P}_{\mathbb{B}}) & \longrightarrow & \boldsymbol{F}(\mathbf{mr}) \\ \downarrow & \cong & \downarrow \\ \mathbf{Set} & \longrightarrow & \boldsymbol{T}(\mathbf{mr}) \end{array} .$$

Appendix

Pre-equipments as double categories

Double categories

Definition

A double category $\mathfrak{C}$ is an internal category in **Cat**, represented by a span

$$\mathbb{C}_0 \xleftarrow{L} \mathbb{C}_1 \xrightarrow{R} \mathbb{C}_0$$

with suitable composition and identity functors.

From a pre-equipment $\mathscr{C}$, we can construct a double category $\tilde{\mathscr{C}}$ as follows:

- $\tilde{\mathscr{C}}$ has the same **objects** as $\mathscr{C}$
- **Horizontal 1-cells** of $\tilde{\mathscr{C}}$ arbitrary 1-cells of $\mathscr{C}$
- **Vertical 1-cells** of $\tilde{\mathscr{C}}$ are regular 1-cells of $\mathscr{C}$
- A **2-cell** $\begin{array}{ccc} A & \xrightarrow{f} & B \\ {\scriptstyle i}\downarrow & \Searrow^{\alpha} & \downarrow{\scriptstyle j} \\ C & \xrightarrow[g]{} & D \end{array}$ in $\tilde{\mathscr{C}}$ is a 2-cell $\begin{array}{ccc} A & \xrightarrow{f} & B \\ {\scriptstyle i}\downarrow & \Swarrow_{\alpha} & \downarrow{\scriptstyle j} \\ C & \xrightarrow[g]{} & D \end{array}$ in $\mathscr{C}$.

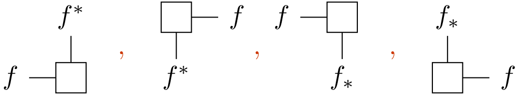
Proarrow equipments

Definition

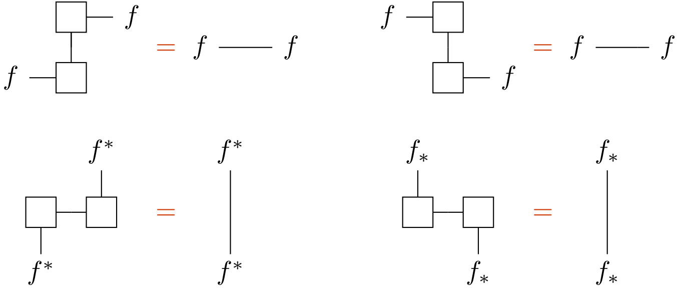
A **proarrow equipment** [7] is a pre-equipment in which each regular 1-cell has a right adjoint.

In *Framed bicategories and monoidal fibrations* [6], Michael Shulman characterized the double categories that can be obtained from equipments, he called them **framed bicategories**.

A **framed bicategory** is a double category $\mathscr{D}$ where for each vertical 1-cell $f : A \to B$ there exist horizontal 1-cells $f^* : A \to B$ and $f_* : B \to A$ and 2-cells



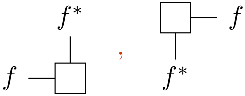
such that



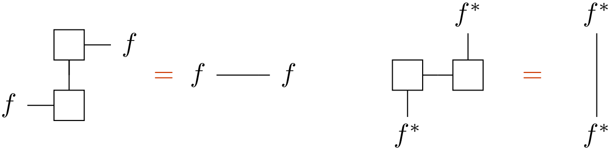
- The arrows f^* were called **companions** by Brown and Mosa in [1].
- f^* and f_* were studied by Grandis and Paré in *Adjoints for double categories* [3].

General pre-equipments as double categories

- The double categories that we obtain as $\tilde{\mathscr{C}}$ from general pre-equipments (without adjoints) are precisely those, where for every horizontal f there exists f^* (and not necessarily f_*).



such that



- In the following, we will call such a double category a **semi-framed bicategory**

Double functors and double transformations

- An **oplax double functor** F between double categories $\mathfrak{C} = \mathbb{C}_0 \xleftarrow{L} \mathbb{C}_1 \xrightarrow{R} \mathbb{C}_0$ and $\mathfrak{D} = \mathbb{D}_0 \xleftarrow{L} \mathbb{D}_1 \xrightarrow{R} \mathbb{D}_0$ is given by a pair of functors

$$F_0 : \mathbb{C}_0 \to \mathbb{D}_0 \quad \text{and} \quad F_1 : \mathbb{C}_1 \to \mathbb{D}_1$$

  and natural families of 2-cells

$$\begin{array}{ll} F(g \circ f) \to Fg \circ Ff & f, g \text{ horizontal 1-cells} \\ F(U_C) \to U_{FC} & U_C, U_{FC} \text{ horizontal identities} \end{array}$$

  subject to the usual coherence conditions.

- A double transformation $\eta : F \to G : \mathfrak{C} \to \mathfrak{D}$ between double functors F, G is given by
  - for each $C \in \mathfrak{C}$ a **vertical 1-cell** $\eta_C : FC \to GC$, and
  - for each **horizontal 1-cell** $f : C \to D$ in $\mathfrak{C}$ a 2-cell $\begin{array}{ccc} FC & \xrightarrow{Ff} & FD \\ {\scriptstyle\eta_C}\downarrow & \searrow_{\eta_f} & \downarrow{\scriptstyle\eta_D} \\ GC & \xrightarrow[Gf]{} & GD \end{array}$,

  subject to coherence conditions.

Morphisms of pre-equipments as oplax double functors

- Morphisms of pre-equipments give rise to oplax double functors between the corresponding semi-framed bicategories.
- However, not every oplax double functor between semi-framed bicategories arises this way.
- Morphimsms of equipments correspond to **normal** oplax double functors.
- Transformations of equipments correspond to double transformations

Summing up

- We observe that the seemingly ad hoc concepts of morphism and transformation of equipments arise naturally in the context of double categories.
- Slogan:

  Double categories are the natural environment of (op)lax functors and transformations.
- This idea comes apparently from from the thesis of Verity and from the the work of Dawson, Paré, Pronk. It was subsequently promoted by Michael Shulman.

Thanks for your attention!

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