Coherent families of Hermitian adjunctions

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Review: Involutive monoidal categories

Definition: Monoidal category $(\mathcal{K}, i, \otimes)$ plus functor $\mathcal{K} \xrightarrow{\overline{(\;)}} \mathcal{K}$ and natural isos $\overline{p} \otimes \overline{q} \xrightarrow{\chi_{p,q}} \overline{q \otimes p}$, $\overline{\overline{r}} \xrightarrow{\varepsilon_r} r$ satisfying

$$\begin{tikzcd}
(\overline{p} \otimes \overline{q}) \otimes \overline{r} \arrow[r, "\alpha_{\overline{p},\overline{q},\overline{r}}"] \arrow[d, "\chi_{p,q} \otimes \mathsf{id}_{\overline{r}}"'] & \overline{p} \otimes (\overline{q} \otimes \overline{r}) \arrow[d, "\mathsf{id}_{\overline{p}} \otimes \chi_{q,r}"] \\
\overline{q \otimes p} \otimes \overline{r} \arrow[d, "\chi_{q \otimes p, r}"'] & \overline{p} \otimes \overline{r \otimes q} \arrow[d, "\chi_{p, r \otimes q}"] \\
\overline{p \otimes (q \otimes r)} & \overline{(r \otimes q) \otimes p} \arrow[l, "\overline{\alpha_{r,q,p}}"]
\end{tikzcd}
\qquad
\begin{tikzcd}
\overline{\overline{p}} \otimes \overline{\overline{q}} \arrow[r, "\chi_{\overline{p},\overline{q}}"] \arrow[d, "\varepsilon_p \otimes \varepsilon_q"'] & \overline{\overline{q} \otimes \overline{p}} \arrow[d, "\overline{\chi_{q,p}}"] \\
p \otimes q & \overline{\overline{p \otimes q}} \arrow[l, "\varepsilon_{p \otimes q}"]
\end{tikzcd}$$

$$\overline{\overline{\overline{r}}} \xrightarrow{\overline{\varepsilon_r} = \varepsilon_{\overline{r}}} \overline{r}$$

Note: Beggs & Majid [2008] study more general concept (*bar monoidal categories*) lacking square.

Examples:

- $(\mathbf{Cat}, \times, 1, (\ )^{\mathsf{op}})$, $(\mathbf{Cat}, +_{lex}, 0, (\ )^{\mathsf{op}})$, $(\mathbf{Flo}, +_{lex}, 0, (\ )^{\mathsf{op}})$;

- $(\mathbf{Vec}, \otimes, \mathbb{C}, \overline{(\ )})$, $(\mathbf{Ban}, \widehat{\otimes}, \mathbb{C}, \overline{(\ )})$, $(\mathbf{Ban}, \check{\otimes}, \mathbb{C}, \overline{(\ )})$;

- $(\mathbf{Oper}, \widehat{\otimes}, \mathbb{C}, \widetilde{(\ )})$, $(\mathbf{Oper}, \boxtimes, \mathbb{C}, \widetilde{(\ )})$, $(\mathbf{Oper}, \check{\otimes}, \mathbb{C}, \widetilde{(\ )})$, $\widetilde{p} := \overline{p}^{\mathsf{op}}$.

More generally, $({}_k\mathbf{Mod}_k, \otimes_k, k)$ can be made involutive whenever k is involutive.

Lemma: $\exists! i \xrightarrow{\eta} \bar{i}$ satisfying

$$\begin{tikzcd} i \otimes \overline{q} \arrow[r, "\eta \otimes \mathsf{id}_q"] \arrow[d, "\lambda_{\overline{q}}"'] & \bar{i} \otimes \overline{q} \arrow[d, "\chi_{i,q}"] \\ \overline{q} & \overline{q \otimes i} \arrow[l, "\overline{\rho_q}"] \end{tikzcd} \qquad \begin{tikzcd} i \arrow[r, "\eta"] \arrow[d, equal] & \bar{i} \arrow[d, "\overline{\eta}"] \\ i & \bar{\bar{i}} \arrow[l, "\varepsilon_i"] \end{tikzcd} \qquad \begin{tikzcd} \overline{q} \otimes i \arrow[r, "\mathsf{id}_{\overline{q}} \otimes \eta"] \arrow[d, "\rho_{\overline{q}}"'] & \overline{q} \otimes \bar{i} \arrow[d, "\chi_{q,i}"] \\ \overline{q} & \overline{i \otimes q} \arrow[l, "\overline{\lambda_q}"] \end{tikzcd}$$

Theorem:

$$(\mathcal{K}, i, \otimes^{\mathsf{rev}}) \underset{(\overline{(\ )}, \eta, \chi^{\mathsf{rev}})}{\overset{(\overline{(\ )}, \eta, \chi)}{\rightleftarrows}} (\mathcal{K}, i, \otimes)$$

is a monoidal adjunction, with ε^{-1} for unit, and ε for counit.

Hermitian adjunctions

Definition: An *involutive object* is a pair $(d, \overline{d} \xrightarrow{\sigma} d)$ satisfying

$$\begin{tikzcd} \overline{\overline{d}} \arrow[r, "\overline{\sigma}"] \arrow[rr, bend right, "\varepsilon_d"'] & \overline{d} \arrow[r, "\sigma"] & d. \end{tikzcd}$$

Examples: $(\mathbb{C}, \overline{(\ )})$, $(\mathbb{C}, -\overline{(\ )})$ in relevant cases; (i, η^{-1}) always.

Definition: A (d-valued) sesquilinear form $\overline{p} \otimes p \xrightarrow{\gamma} d$ is called (σ-)*Hermitian* if

$$\begin{tikzcd} \overline{p} \otimes \overline{\overline{p}} \arrow[r, "\sim"] \arrow[d, "\sim"'] & \overline{\overline{p} \otimes p} \arrow[r, "\overline{\gamma p}"] & \overline{d} \arrow[d, "\sigma"] \\ \overline{p} \otimes p \arrow[rr, "\gamma p"] & & d. \end{tikzcd}$$

Convention: $(d, \sigma) = (i, \eta^{-1})$ unless otherwise specified.

Definition: A *Hermitian adjunction* is a pair of arrows

$$i \xrightarrow{\tau} p \otimes \overline{p} \qquad\qquad \overline{p} \otimes p \xrightarrow{\gamma} i$$

satisfying usual triangle identities, and such that γ is Hermitian.

Example: A HA in $(\mathbf{Vec}, \otimes, \mathbb{C})$ is a finite dimensional space plus an *indefinite* inner product.

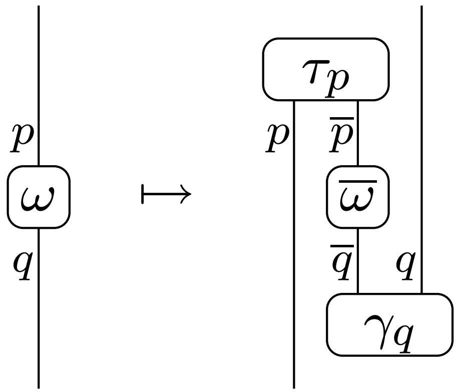
Definition: A *family of HAs* is a choice of HA for every object.

$$i \xrightarrow{\tau_p} p \otimes \overline{p} \qquad\qquad \overline{p} \otimes p \xrightarrow{\gamma_p} i$$

Note: We do *not* assume naturality!

Daggers

Lemma: Given an IMC plus a family of HAs, then



defines an identity-on-objects contravariant involution—*alias*, a *dagger* operation—on the underlying (mere) category.

Lemma: The equations $(\psi \otimes \omega)^\dagger = \psi^\dagger \otimes \omega^\dagger$, $\alpha^\dagger = \alpha^{-1}$, $\lambda^\dagger = \lambda^{-1}$ and $\rho^\dagger = \rho^{-1}$ hold—*i.e.*, $(\mathcal{K}, \otimes, i, (\ )^\dagger)$ is *dagger monoidal*—iff

$$\begin{tikzcd}
(\overline{q} \otimes \overline{p}) \otimes (p \otimes q) \arrow[r, "\sim"] \arrow[d, "\sim"'] & \overline{p \otimes q} \otimes (p \otimes q) \arrow[d, "\gamma_{p \otimes q}"] \\
\overline{q} \otimes ((\overline{p} \otimes p) \otimes q) \arrow[d, "\mathrm{id}_{\overline{q}} \otimes (\gamma_p \otimes \mathrm{id}_q)"'] & i \\
\overline{q} \otimes (i \otimes q) \arrow[r, "\sim"'] & \overline{q} \otimes q \arrow[u, "\gamma_q"']
\end{tikzcd}$$

and
$$\begin{tikzcd}
i \otimes i \arrow[r, "\sim"] \arrow[rr, bend right, "\sim"'] & \overline{i} \otimes i \arrow[r, "\gamma_i"] & i
\end{tikzcd}$$
hold.

Note: These diagrams are intuitive:

- $\langle a, b \rangle = \overline{a} \cdot b$ should hold for scalars, and
- $\langle u \otimes x, v \otimes y \rangle = \langle x, \langle u, v \rangle \cdot y \rangle$ should hold for pure tensors.

  $(= \langle u, v \rangle \cdot \langle x, y \rangle\,,$ classically)

Cyclic involutive monoidal categories

Definition: [Maltsiniotis] A *sovereign (s,t)-structure* on a monoidal category $(\mathcal{K}, \otimes, i)$ is a pair of natural isomorphisms

$$\langle p \otimes t, i\rangle_{\mathcal{K}} \xrightarrow{\widehat{\Phi}} \langle t \otimes p, i\rangle_{\mathcal{K}}$$

$$\langle i, t \otimes p\rangle_{\mathcal{K}} \xrightarrow{\check{\Phi}} \langle i, p \otimes t\rangle_{\mathcal{K}}$$

satisfying a number of coherence conditions.

We shall use the term *cyclic structure* for this concept.

Example: If β is a symmetry, then composing with β yields a cyclic structure. (The same is not true of braids.)

Definitions: A *CIMC* is an IMC together with a cyclic structure satisfying

$$
\begin{array}{ccc}
\langle p \otimes t, i\rangle_{\mathcal{K}} & \xrightarrow{\widehat{\Phi}} & \langle t \otimes p, i\rangle_{\mathcal{K}} \\
\chi\,;\,\overline{(\;)}\,;\,\eta^{-1} \big\downarrow & & \big\downarrow \chi\,;\,\overline{(\;)}\,;\,\eta^{-1} \\
\langle \overline{t} \otimes \overline{p}, i\rangle_{\mathcal{K}} & \xrightarrow{\widehat{\Phi}} & \langle \overline{p} \otimes \overline{t}, i\rangle_{\mathcal{K}}
\end{array}
\qquad
\begin{array}{ccc}
\langle i, t \otimes p\rangle_{\mathcal{K}} & \xrightarrow{\check{\Phi}} & \langle i, p \otimes t\rangle_{\mathcal{K}} \\
\eta\,;\,\overline{(\;)}\,;\,\chi^{-1} \big\downarrow & & \big\downarrow \eta\,;\,\overline{(\;)}\,;\,\chi^{-1} \\
\langle i, \overline{p} \otimes \overline{t}\rangle_{\mathcal{K}} & \xrightarrow{\check{\Phi}} & \langle i, \overline{t} \otimes \overline{p}\rangle_{\mathcal{K}}
\end{array}
$$

A *SIMC* (*BIMC*) is an IMC plus a symmetry (braiding) satisfying

$$
\begin{array}{ccc}
\overline{p} \otimes \overline{q} & \xrightarrow{\chi_{p,q}} & \overline{q \otimes p} \\
\beta_{\overline{p},\overline{q}} \big\downarrow & & \big\downarrow \overline{\beta_{q,p}} \\
\overline{q} \otimes \overline{p} & \xrightarrow{\chi_{q,p}} & \overline{p \otimes q}.
\end{array}
$$

Note: Every SIMC (but not every BIMC) is a CIMC.

Theorem: A *dagger pivotal category* is the same thing as a CIMC plus a family of HAs satisfying

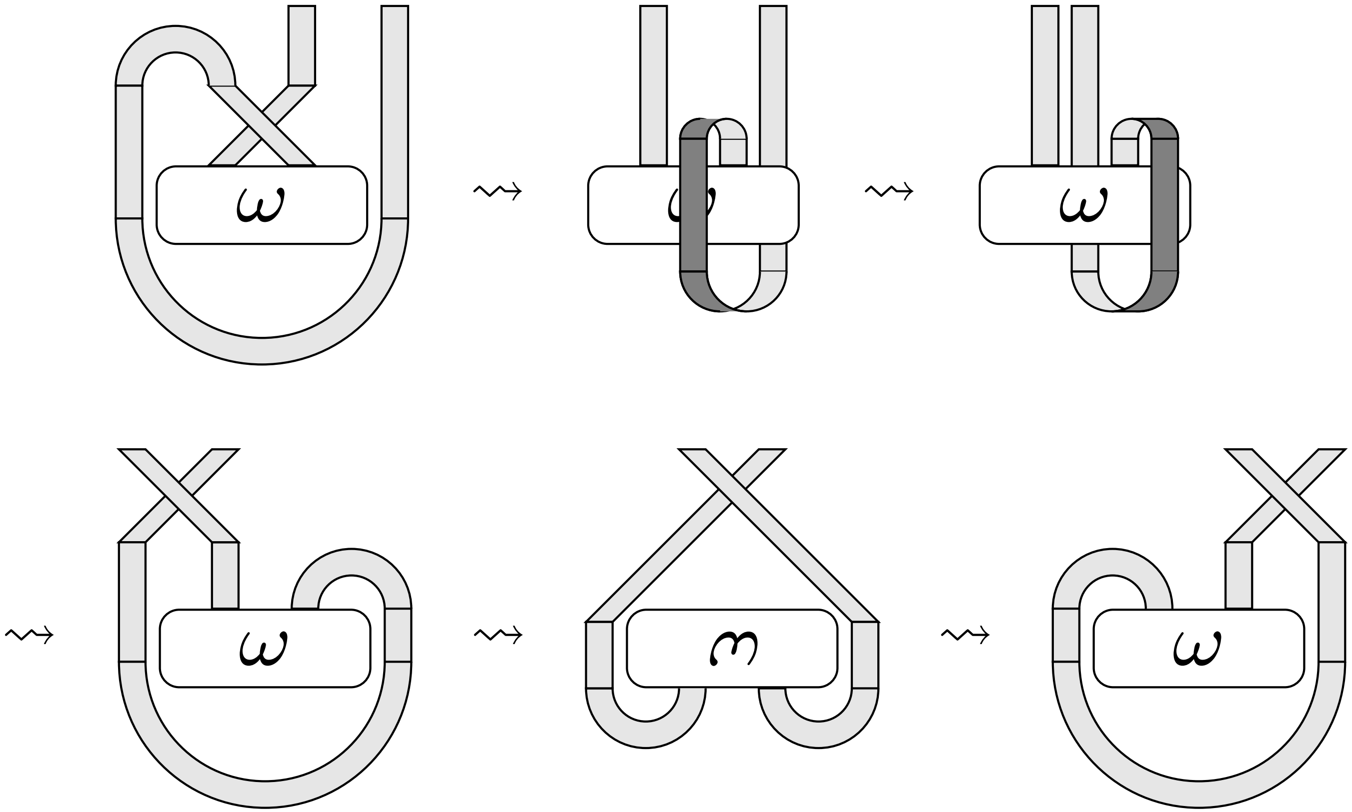
$$\begin{tikzcd}
\overline{\overline{p}} \otimes \overline{p} \arrow[r, "\gamma_{\overline{p}}"] \arrow[d, "\sim"'] & i \\
p \otimes \overline{p} \arrow[ur, "\hat{\Phi}(\gamma_p)"'] &
\end{tikzcd}$$

as well as the previous two diagrams.

A *dagger compact closed category* is the same thing as a SIMC plus a family of HAs satisfying the same axioms.

A *dagger tortile category* is the same thing as a BIMC plus a family of HAs satisfying the same axioms and also

$$\begin{tikzcd}
\langle p \boxtimes q, d \rangle_{\mathcal{K}} \arrow[r, "\Phi_{p,q}"] \arrow[d, "\beta_{q,p}\ ;\ (\ )"'] & \langle q \boxtimes p, d \rangle_{\mathcal{K}} \arrow[d, "\beta_{p,q}\ ;\ (\ )"] \\
\langle q \boxtimes p, d \rangle_{\mathcal{K}} \arrow[r, "\Phi_{q,p}"] & \langle p \boxtimes q, d \rangle_{\mathcal{K}}
\end{tikzcd}$$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