Action accessibility and centralizers

Alan Cigoli
Università degli Studi di Milano
(joint work with Sandra Mantovani)

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The notion of action accessible category
(D. Bourn and G. Janelidze, ’09)

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- Arrows:

$$\begin{array}{ccccc} X & \xrightarrow{x} & A & \underset{s}{\overset{p}{\rightleftarrows}} & B \\ \| & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle g} \\ X & \xrightarrow{k} & C & \underset{t}{\overset{q}{\rightleftarrows}} & D \end{array}$$

with $fx = k$, $fs = tg$ and $qf = gp$.

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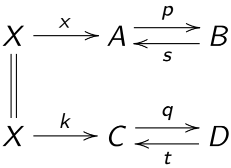

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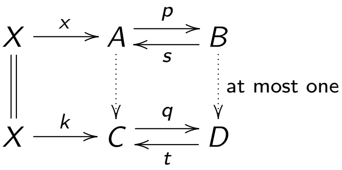

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Definition (D. Bourn and G. Janelidze)

Let $\mathbb{C}$ be a pointed protomodular category. $\mathbb{C}$ is said to be **action accessible** *if, for any $X \in \mathbb{C}$, every object in $\mathrm{SplExt}_{\mathbb{C}}(X)$ admits a morphism into a faithful one.*

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- Categories of interest in the sense of Orzech (A. Montoli), such as Leibniz algebras, Poisson algebras etc.

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Faithful split extensions are easily characterized:

$$X \xrightarrow{x} A \underset{s}{\overset{p}{\rightleftarrows}} B \quad \text{is faithful} \iff Z(X,B) = 0$$

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2. *a split extension in $\mathbb{C}$ is faithful if and only if its centralizer is the zero object.*

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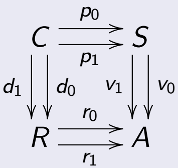
Theorem (D. Bourn and G. Janelidze)

The kernel pair of g is the **centralizer** *of R, that is, the largest equivalence relation on A that commutes with R*

Definition (M.C. Pedicchio)

Let R and S be equivalence relations on A. We'll say that $[R, S] = 0$ if and only if there exists a C, equivalence relation on both R and S, such that, in the diagram below, the four squares where parallel arrows have the same index are pullbacks:

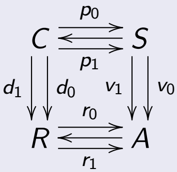


Non symmetric commutativity and ns-centralizers of equivalence relations

Definition (non symmetric)

Let S be a relation on A and R an equivalence relation on A. We'll say that $]S, R] = 0$ if and only if there exists (C, p_0, p_1, t_0) equivalence relation on S with (C, d_0, d_1) relation on R such that

1. *In the diagram below, the four squares where parallel arrows have the same index are pullbacks:*



2. *If $k : X \to C$ is a kernel of p_0 (or p_1 equivalently), then $d_0 k = d_1 k$.*

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Let R and S be equivalence relations on A, then:

$$[S, R] = 0 \iff]S, R] = 0$$

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Given an equivalence relation R on an object A. A **ns-centralizer** *for R is an equivalence relation $E_A(R)$ on A such that:*

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Let $\mathbb{C}$ be a homological action accessible category. Then $\mathbb{C}$ has ns-centralizers for equivalence relations.

A characterization of action accessible categories

$$X \xrightarrow{x} A \underset{s}{\overset{p}{\rightleftarrows}} B$$

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Assume that $R[p]$ has ns-centralizer $E_A(R[p])$, with associated normal subobject Z_A.

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Define:

$$\begin{array}{ccc} E_B & \longrightarrow & E_A \\ {\scriptstyle \langle v_0, v_1 \rangle}\downarrow & \lrcorner & \downarrow{\scriptstyle \langle z_0, z_1 \rangle} \\ B \times B & \xrightarrow{\;s \times s\;} & A \times A \end{array}$$

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3. $Z_B = 0$

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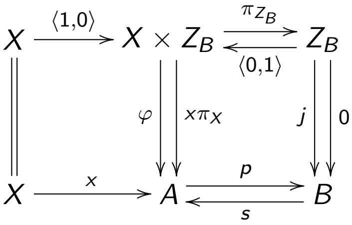

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Suppose $Z_B \neq 0$, then:

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\| & & \varphi \downarrow\downarrow x\pi_X & & j \downarrow\downarrow 0 \\
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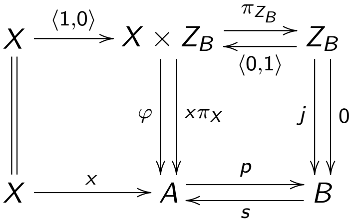

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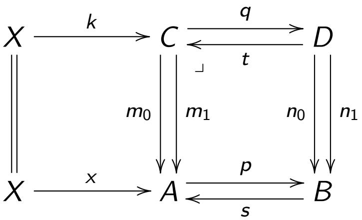

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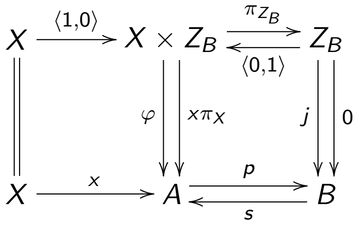

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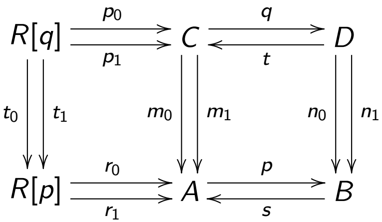

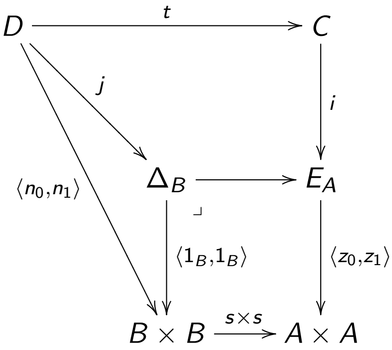
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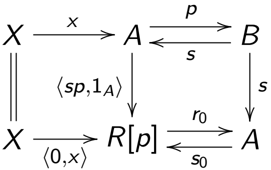

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