The universal loop space operad and generalisations

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University of Sheffield
CT 2010

Plan

1. Introduction

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2. Operad actions

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3. The universal loop space operad

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4. Higher-dimensional generalisation

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4. Higher-dimensional generalisation
5. Non-universal operads.

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- homotopies between paths
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—compare with multiplication in loop spaces.

Aim: To express this structure via an operad action.

Trimble's approach

"Extract one dimension at a time, and iterate"

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So we need to understand operads that act on path spaces.

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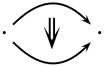
- 2-cells 
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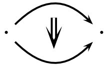
- 2-cells 
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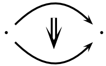
- 2-cells 
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We need to do this for n-categories because we're not so good at the practice part yet...

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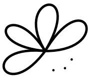

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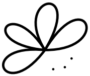
k-ary operation  $\longmapsto$ one loop

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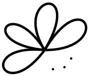

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$$S^{\vee k} \xrightarrow{k \text{ loops}} X$$

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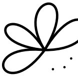

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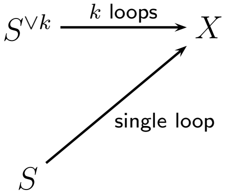

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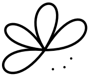
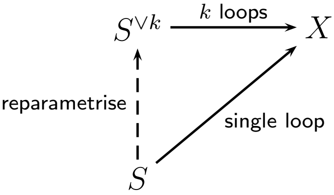
k-ary operation  $\longmapsto$ one loop



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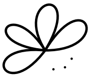

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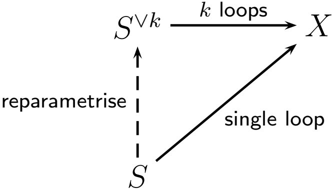

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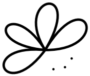
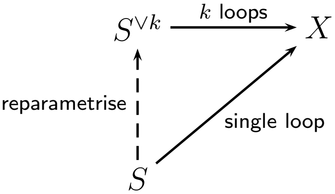
k-ary operation  $\longmapsto$ one loop



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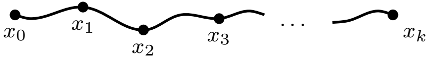

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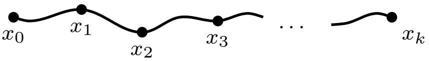

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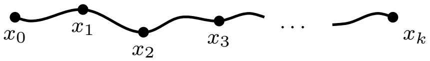

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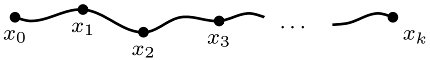

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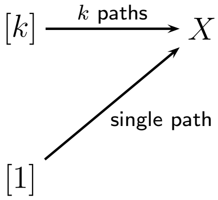

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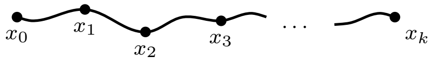
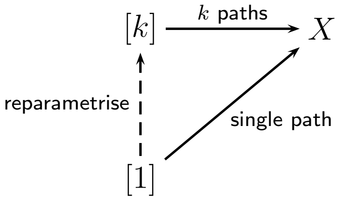
k-ary operation  $\longmapsto$ one path



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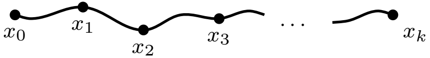

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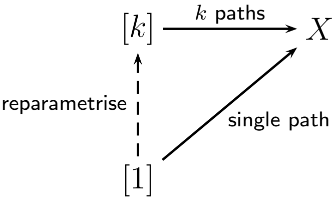

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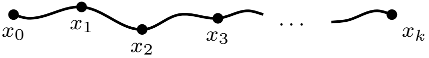

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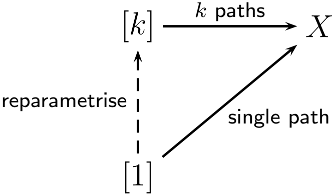

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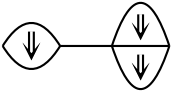

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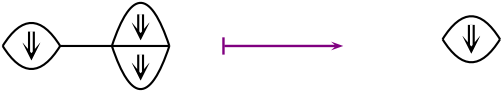

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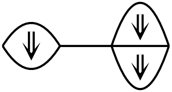

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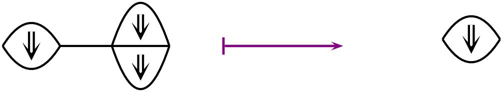

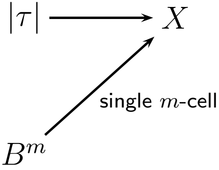

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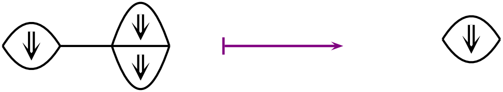

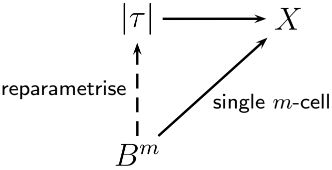

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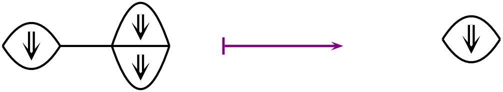

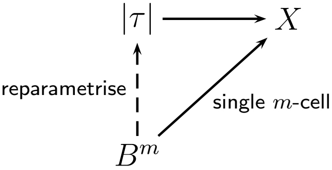

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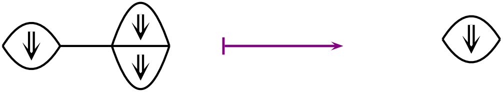

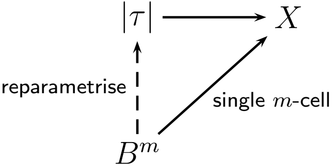

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Full definition

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2. Operad actions

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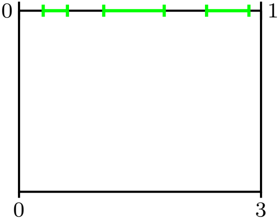

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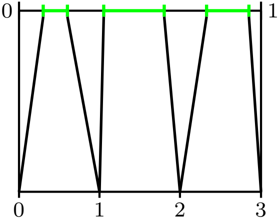

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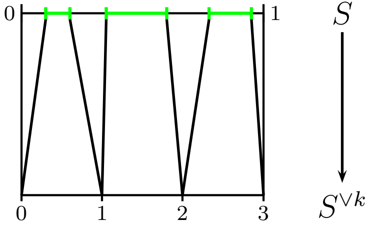

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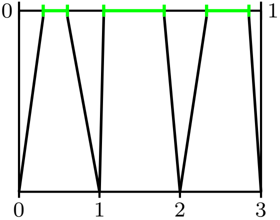
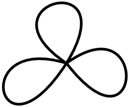


$$\begin{array}{c} S \\ \big\downarrow \\ S^{\vee k} \end{array}$$



$$S^{\vee k} \longrightarrow X$$

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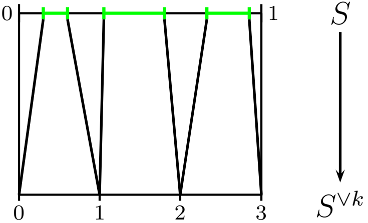

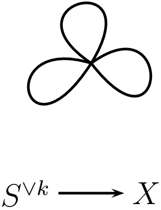

We have made $D(k) \longrightarrow \mathcal{S}(S, S^{\vee k})$

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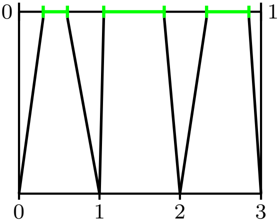

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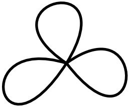

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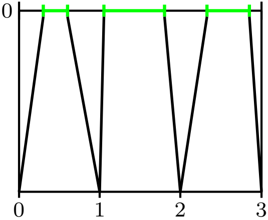

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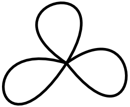

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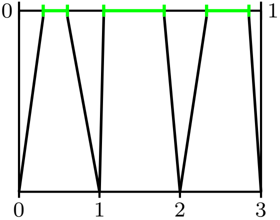

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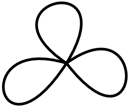

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This clearly also works for an action on path spaces.

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This determines the whole action by naturality.

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Theorem (Salvatore)

Let P be an operad in $\mathbf{Top}_*$.

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A categorical proof enables us to generalise immediately to higher dimensions.

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So $(\alpha_k)_* : P(k) \longrightarrow L(k)$.

An action of P on loop spaces is $\forall k$ a natural transformation

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4. Higher-dimensional generalisation

Operads acting on ω-path spaces

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$$UX_m = \mathbf{Top}(B^m, X).$$

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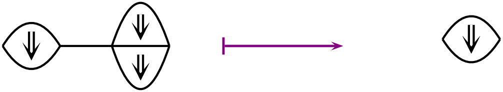

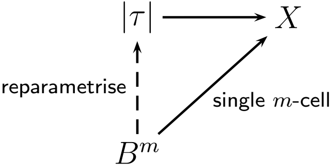

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universal recognition operad	$L = \mathbf{End}(\Omega)(*)$ $L(k) = \mathbf{Top}_*(S, S^{\vee k})$	$G = \mathbf{End}(U)(*)$ $G(\tau) = \mathbf{Top}(B^m, \lvert\tau\rvert)$

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Note that $\mathcal{P}_\omega = \Pi_\omega = \Phi_\omega = U$.

5. Non-universal operads

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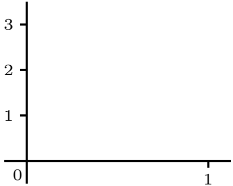
E
universal



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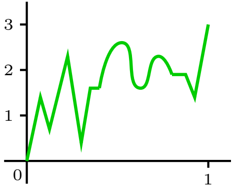

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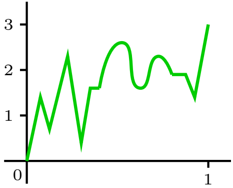
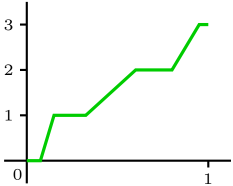


D
little intervals



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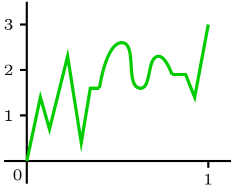

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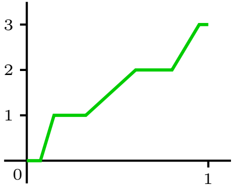
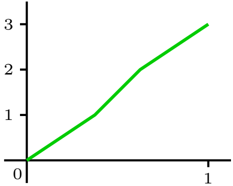


K
Stasheff associahedra



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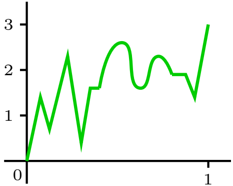

$\uparrow$

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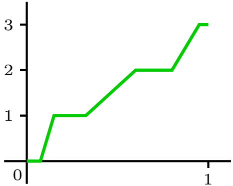

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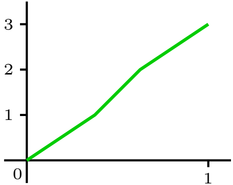

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$$
\begin{array}{ccc}
\text{Trimble } n\text{-categories} & \xrightarrow{\beta} & \text{Batanin } n\text{-categories} \\[2em]
\begin{array}{c}\text{classical operad}\\ \text{acting on path spaces}\end{array} & \longmapsto & \begin{array}{c}\text{globular operad}\\ \text{acting on } \omega\text{-path spaces}\end{array} \\[2em]
\text{universal } E & & \text{universal } G \\[2em]
E & \longmapsto & \beta(E) \hookrightarrow G \\
\uparrow & & \uparrow \\
\text{little intervals } D & & \beta(D) \\
\uparrow & & \uparrow \\
\text{Stasheff } K & & \beta(K)
\end{array}
$$