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Torsors, Herds and Flocks

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Classically a herd is a set T together with a ternary operation

$$q : T \times T \times T \to T$$

such that the following axioms hold:

$$\begin{aligned} q(q(a,b,c),d,e) &= q(a,b,q(c,d,e)) & (1)\\ q(a,b,b) &= a & (2)\\ q(a,a,b) &= b\,. & (3) \end{aligned}$$

If $T = G$ is a group then defining $q(x,y,z) = xy^{-1}z$ gives G as a herd.

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Let $\mathbb{V}$ be a nice braided monoidal category. A comonoid $A = (A, \delta, \epsilon)$ in $\mathbb{V}$ is defined to be a herd when it is equipped with a comonoid morphism

$$q : A \otimes A^o \otimes A \to A$$

such that the analogous axioms hold:

$$A^{\otimes 5} \xrightarrow{q \otimes 1 \otimes 1} A^{\otimes 3} \xrightarrow{q} A \;=\; A^{\otimes 5} \xrightarrow{1 \otimes 1 \otimes q} A^{\otimes 3} \xrightarrow{q} A$$

$$A \otimes A \xrightarrow{1 \otimes \delta} A \otimes A \otimes A \xrightarrow{q} A \;=\; A \otimes A \xrightarrow{1 \otimes \epsilon} A$$

$$A \otimes A \xrightarrow{\delta \otimes 1} A \otimes A \otimes A \xrightarrow{q} A \;=\; A \otimes A \xrightarrow{\epsilon \otimes 1} A$$

Let G be a monoid in the category of finite sets **Set** (or any topos). A G-torsor is a set A (or object) together with a G-action

$$\mu : G \times A \longrightarrow A$$

such that:

- the unique map $! : A \rightarrow \{*\}$ is a regular epimorphism
- the morphism $(\mu, 1) : G \times A \longrightarrow A \times A$ is invertible
- The first requirement saying that A is non-empty is what we wish to generalize.

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For any comonoid A in $\mathbb{V}$ we take the category $Comod_\ell(A)$ of left A-comodules. We then define the monad $T_A : Comod_\ell(A) \to Comod_\ell(A)$ by tensoring on the left by A:

$$T_A(M \to A \otimes M) = (A \otimes M \to A \otimes A \otimes M)$$

with the components of the multiplications and unit being

$$1 \otimes \epsilon \otimes 1 : A \otimes A \otimes M \to A \otimes M, \text{ and } \delta : M \to A \otimes M$$

and comparison functor

$$\kappa : \mathbb{V} \to (Comod_\ell(A))^{T_A}.$$

When this κ is fully faithful then we say that the morphism $(A \to I)$, where I is the tensor unit, is a codescent morphism. This is the generalization that we need.

A torsor in $\mathbb{V}$ is a herd A for which the counit $\epsilon : A \longrightarrow I$ is a codescent morphism.

- It turns out that each torsor A is a left H-torsor for a Hopf monoid H.
- If A is only a herd then symmetrically we find that A becomes a right H'-torsor for a Hopf monoid H'.

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We wish to lift the definition of a herd up to a higher dimension. A 2-dimensional (or multi-objected) herd could be called a ‘bi-herd’. This conveniently contracts to ‘bird’ and so they became known as Flocks.

A $\mathbb{V}$-category $\mathcal{A}$ is called a $\mathbb{V}$-flock when it is equipped with a $\mathbb{V}$-functor

$$Q : \mathcal{A} \otimes \mathcal{A}^{\mathrm{op}} \otimes \mathcal{A} \to \mathcal{A}$$

and $\mathbb{V}$-natural transformations

$$\phi : Q(Q(A, B, C), D, E) \xrightarrow{\cong} Q(A, B, Q(C, D, E))$$

$$\alpha : Q(A, B, B) \longrightarrow A$$

$$\beta : B \longrightarrow Q(A, A, B)$$

which then satisfy three rather typical axioms.

As an example let $\mathbb{V}_f$ be the full subcategory of $\mathbb{V}$ for which every object has a dual (e.g. $\mathbb{V} = \mathbf{Vect_k}$). If we let $Q(A, B, C) = A \otimes B^* \otimes C$ then the ϕ is simply given by the associativity isomorphism for the tensor product. There are also quite natural choices for α and β:

$$\alpha : A \otimes B^* \otimes B \xrightarrow{1 \otimes \epsilon} A$$

$$\beta : B \xrightarrow{\eta \otimes 1} A \otimes A^* \otimes B \ .$$

The axioms are then easy to check; they are just the "triangles" for the unit and counit of the duals in question. One could call this the fundamental example for reasons soon to be apparent.

This last section here summarizes the last two sections of the paper to be published with the same name as the title of this talk.
The basic idea is that one should be able to take a herd and lift it up to a flock and, vice versa, one should be able to take a flock and drop it down to a herd.

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Let $\mathbb{V}$ be a left autonomous monoidal category and A be a herd in $\mathbb{V}$ as defined before. Let $\mathcal{A} = Comod_f(A)$ the category of right A-comodules whose underlying objects in $\mathbb{V}$ have duals. We define the Q, ϕ, α, and β to be the same as for $\mathbb{V}_f$ where one should note that M^* means the dual comodule.

We recall the forgetful functor $U : \mathcal{A} \to \mathbb{V}_f$ (forgets the A-comodule structure). As is most often the case, U is faithful (and a little bit more...). Thus, if $Q(L, M, N)$ can be given an A-comodule structure and ϕ, α, and β are A-comodule morphisms, then the axioms for a $\mathbb{V}$-flock that hold in $\mathbb{V}_f$ lift directly to give a $\mathbb{V}$-flock structure on $\mathcal{A}$.

Indeed this is exactly what happens. It is precisely the defining property of A being a herd (the existence of a q) that allows us to give $Q(L, M, N)$ an A-comodule structure.

There is a sensible notion of functor between flocks.

Thus, let $\mathcal{A}$ and $\mathcal{X}$ be $\mathbb{V}$-flocks. A $\mathbb{V}$-functor $F : \mathcal{A} \to \mathcal{X}$ is said to be flockular when it is equipped with a $\mathbb{V}$-natural family of morphisms

$$\rho_{A,B,C} : Q(FA, FB, FC) \to FQ(A, B, C)$$

such that they satisfy the afore mentioned sensible requirements. We call F strong flockular when each $\rho_{A,B,C}$ is invertible.

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Let $\mathcal{A}$ be a flock as defined before. Suppose that $F : \mathcal{A} \to \mathbb{V}_f$ is strong flockular where $\mathbb{V}_f$ has the flock structure given in the previous section. We claim that when the coend

$$E = End^{\vee}(F) = \int^{A \in \mathcal{A}} (FA)^* \otimes FA$$

exists in $\mathbb{V}$ then E is a herd (in $\mathbb{V}$).

It is known that E is a comonoid. We can obtain the $q : E \otimes E^o \otimes E \longrightarrow E$ in the definition of a herd by using the usual coprojection arguments together with the strong flockularity of F.

The way in which this result and the one for herds-to-flocks falls out of the ideas presented here is quite nice.

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Grazie!