Higher homotopy operations
and
Higher categories

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Higher categories and homotopy theories

A *homotopy theory* is an $(\infty, 1)$-category such as:

- Simplicial categories (enriched in $\mathcal{S}$ = simp. sets)
- Cubical categories (enriched in $\mathcal{C}$ = cubical sets)
- Complete Segal spaces
- Segal categories
- Quasi-categories

Idea (Dwyer-Kan): these encode all the homotopy invariants of a model category $\mathcal{M}$ (beyond those contained in $\operatorname{ho}\mathcal{M}$).
$\Rightarrow$ This is a slogan, not a theorem!

We shall concentrate on the first two examples, because they have truncated versions:

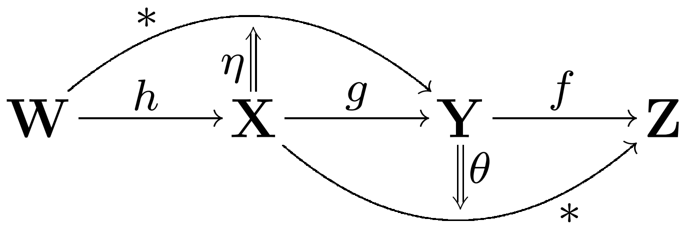
Def: An n-*homotopy theory* is an $(n, 1)$-category $\mathcal{N}$, in which each mapping space is an n-Postnikov section.

Goal: To find enough natural homotopy invariants of $\mathcal{M}$ to determine it up to weak equivalence.

Toda brackets:

Idea: Higher homotopy operations appear as obstructions to rectifying homotopy-commutative diagrams in a model category $\mathcal{M}$.

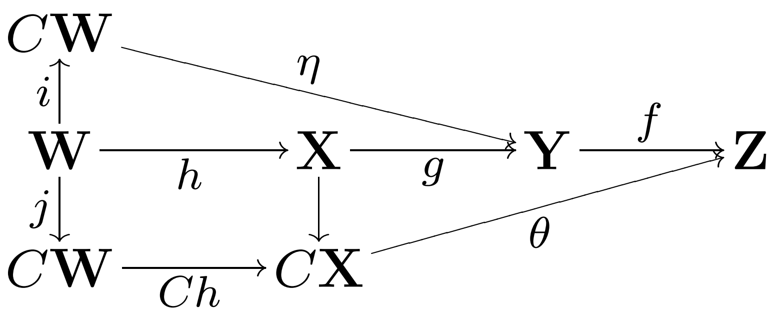
Example:



(1)

Goal: To choose representatives f, g, & h (in their homotopy classes) so diagram commutes on the nose.

Def: The map $\Sigma\mathbf{W} \to \mathbf{X}$ described by



is the *Toda bracket* $\{f, g, h\}$

Warning: Varying η, θ causes indeterminacy. Thus $\{f, g, h\}$ is a subset of $[\Sigma\mathbf{W},\ \mathbf{X}]$.

∞-Homotopy commutative diagrams

Assume: $\mathcal{M}$ is a cubically enriched model category, Γ a indexing category, and $X : \Gamma \to \mathrm{ho}\,\mathcal{M}$ a (homotopy-commutative) diagram, which we want to lift to $\widehat{X} : \Gamma \to \mathcal{M}$.

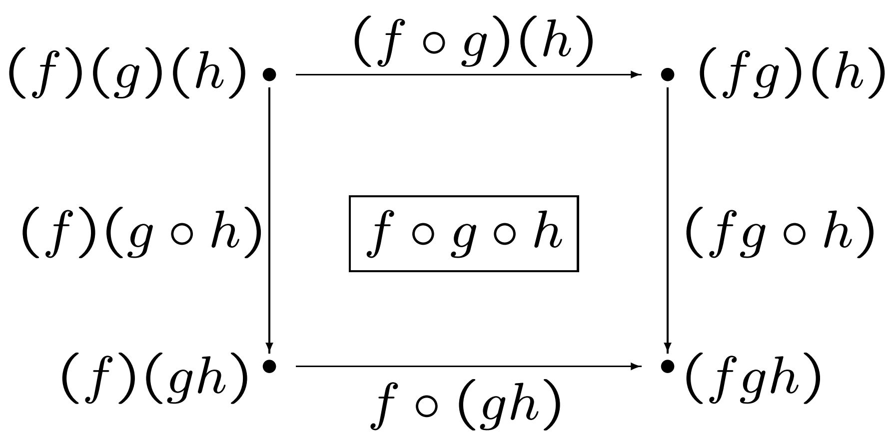
Idea: Use Boardman-Vogt W-construction to replace Γ by a cofibrant cubical model $W\Gamma$.

Note: The usual free simplicial resolution $F\Gamma$ of the category Γ is a triangulation of $W\Gamma$.

Example: For linear Γ:

$$\mathbf{3} \xrightarrow{h} \mathbf{2} \xrightarrow{g} (\mathbf{1}) \xrightarrow{f} \mathbf{0}$$

$W\Gamma$ is a square :



Higher homotopy operations:

Note: Representatives for each $X(\phi)$ (ϕ in Γ) determine a cubical map $X_0 : \mathrm{sk}_0 W\Gamma \to \mathcal{M}$. Try to extend inductively to $X_n : \mathrm{sk}_n W\Gamma \to \mathcal{M}$.

Def: An ∞-*commutative diagram* is any cubical functor extending such an X_0 to $X_\infty : W\Gamma \to \mathcal{M}$.

Theorem (Boardman-Vogt): any X_∞ yields a rectification $\hat{X} : \Gamma \to \mathcal{M}$ of $X : \Gamma \to \mathrm{ho}\,\mathcal{M}$.

Assume: Γ is a *lattice* (=directed, with intitial v_{init} and final v_{fin}) of length $n+1$, so $W\Gamma$ is n-dimensional.

Def (B.-Markl, B.-Chachólski): The *last* obstruction to extending $X_{n-1} : \mathrm{sk}_{n-1} W\Gamma \to \mathcal{M}$ to all $W\Gamma$ reduces to a homotopy class

$$\langle\langle\Gamma\rangle\rangle \in [\Sigma^{n-1} X(v_{\mathrm{init}}),\ X(v_{\mathrm{fin}})]$$

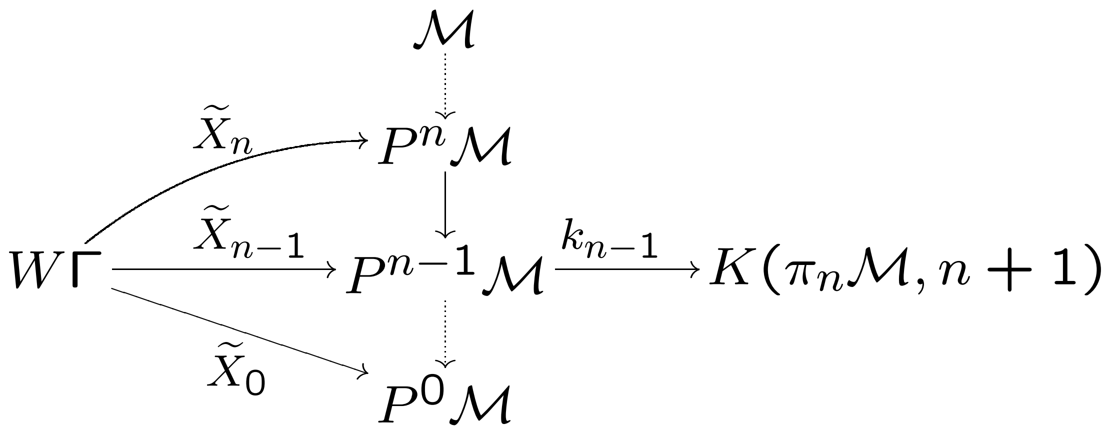
called the n*-th order homotopy operation* associated to X.

Fact: $\langle\langle\Gamma\rangle\rangle$ vanishes (can be chosen to equal 0) if and only if X can be rectified.

Example: Compositions in (1) can be made 0 if and only if the Toda bracket $\{f, g, h\}$ vanishes.

Higher operations and $(n, 1)$-categories:

Note: The right adjoint csk_n to sk_n (on $\mathcal{C}$ or $\mathcal{S}$) is the $(n-1)$-Postnikov section (for fibrant complexes). Thus the previous extension problem is equivalent to a lifting problem:



Fact: The n-th *Dwyer-Kan-Smith obstruction* to rectifying X is the $(\mathcal{C}, \mathcal{O})$-cohomology class

$$[k_{n-1} \circ \widetilde{X}_{n-1}] \in H^{n+1}(\Gamma; \pi_n\mathcal{M}) .$$

Theorem (B.-Johnson-Turner): There is a one-to-one correspondence between such cohomology classes and the higher homotopy operation associated to X.

Remark: All n-th order homotopy operations in a (cubical or simplicial) model category $\mathcal{M}$ depend only on the $(n-1)$-th Postnikov section of $\mathcal{M}$.

Universal invariants:

We have seen how certain homotopy invariants of a model category $\mathcal{M}$ are encoded in the cubical or simplicial enrichment, and that there is a filtration on these invariants corresponding to the Postnikov sections of mapping spaces of $\mathcal{M}$.

Question: Can this be reversed? That is, are there homotopy invariants of $\mathcal{M}$ which suffice to determine its homotopy theory (up to Dwyer-Kan equivalence)?

Note: One could use the k-invariants for $\mathcal{M}$, taking value in its $(\mathcal{S}, \mathcal{O})$-cohomology.

Problem: These are hard to compute, and have only global information about $\mathcal{M}$ as a whole: we need to extract explicit homotopy invariants from them.

Two approaches:

- Find "algebraic" models for n-Postnikov sections of $\mathcal{M}$, and use these to describe k-invariants explicitly.

- Restrict attention to "small parts" of homotopy theories.

Track categories:

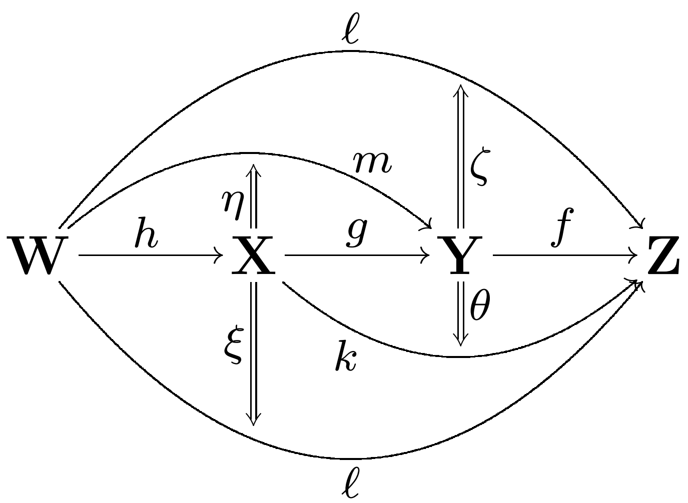
The fundamental groupoid $\hat{\pi}_1(X)$ of a space X models its 1-type, so *track categories* (=categories enriched in groupoids) model $P^1\mathcal{M}$ (as $\mathrm{ho}\,\mathcal{M} = \pi_0\mathcal{M}$ models $P^0\mathcal{M}$).

Invariants for track categories: Each track category $\mathcal{D}$ has $\chi_{\mathcal{D}} \in H^3(\mathrm{ho}\,\mathcal{D}; \pi_1\mathcal{D})$ (Baues-Wirsching cohomology) determining $\mathcal{D}$ up to weak equivalence.

$\chi_{\mathcal{D}}$ assigns to a 3-simplex in the nerve of $\mathrm{ho}\,\mathcal{D}$:

$$\mathbf{W} \xrightarrow{[h]} \mathbf{X} \xrightarrow{[g]} \mathbf{Y} \xrightarrow{[f]} \mathbf{Z}$$

the class in $\pi_1\,\mathrm{map}(\mathbf{W},\mathbf{Z}) = [\Sigma\mathbf{W},\mathbf{Z}]$ corresponding to the self-homotopy of $[\ell] = [f] \circ [g] \circ [h]$:



(2)

Warning: This looks like a Toda bracket, but has *complete* indeterminacy unless $\ell = *$ (otherwise choose $m := gh$, $k := fg$, $\ell := fgh$).

Higher track categories:

$\exists$ various concepts of 2-fundamental groupoids for topological spaces: homotopy double groupoids (Brown-Hardie-Kamps-Porter), homotopy bigroupoids (Hardie-Kamps-Kieboom), strict 2-groupoids (Moerdijk-Svensson), and weak 2-groupoids (Tamsamani).

Fact (B.-Paoli): There is a notion of *two-typical* double groupoids which model 2-types, with a cartesian monoidal structure.

Categories enriched in them, called 2-*track categories*, thus model 2-Postnikov sections of simplicially enriched categories.
Moreover:

Theorem (B.-Paoli): There is a notion of Baues-Wirsching cohomology for track categories.
For each 2-track category $\mathcal{G}$, there is a class $\chi_{\mathcal{G}} \in H^4(\widehat{\pi}_1\mathcal{G}; \pi_2\mathcal{G})$ determining $\mathcal{G}$ up to weak equivalence.

This is described by a 4-dimensional analogue of (2).

Idea: It should be possible to extend these to results to all n (work in progress).

Mapping algebras:

Def: Fix A in a pointed simplicial model category $\mathcal{M}$, and let Θ_A = full sub-simplicial category of generated by A under suspensions and coproducts of cardinality $\leq \lambda$.
An A-*mapping algebra* is a simplicial functor $\mathfrak{X} : \Theta_A^{\text{op}} \to \mathcal{S}_*$. It is *realistic* if it preserves homotopy limits.

Example: Let $A = S^1$ in $\mathcal{M} = \mathcal{T}op_*$, fix $Y \in \mathcal{M}$, and set $\mathfrak{X}\{B\} := \text{map}_{\mathcal{M}}(B, Y)$ (for $B \in \Theta_A$). This is the *realizable* A-mapping algebra $\mathfrak{M}_{\mathcal{A}}Y$, determined by products of iterated loop spaces of $\Omega Y = \mathfrak{M}_{\mathcal{A}}Y\{S^1\}$, equipped with an action of Θ_A by precomposition.

Def: Let $\Theta^{\mathcal{A}}$ be the sub-simplicial category of $\mathcal{M}$ containing $\mathcal{A}$, closed under loops and products of cardinality $\leq \lambda$.
A *dual* $\mathcal{A}$-*mapping algebra* is a simplicial functor $\mathfrak{X} : \Theta_A \to \mathcal{S}_*$.

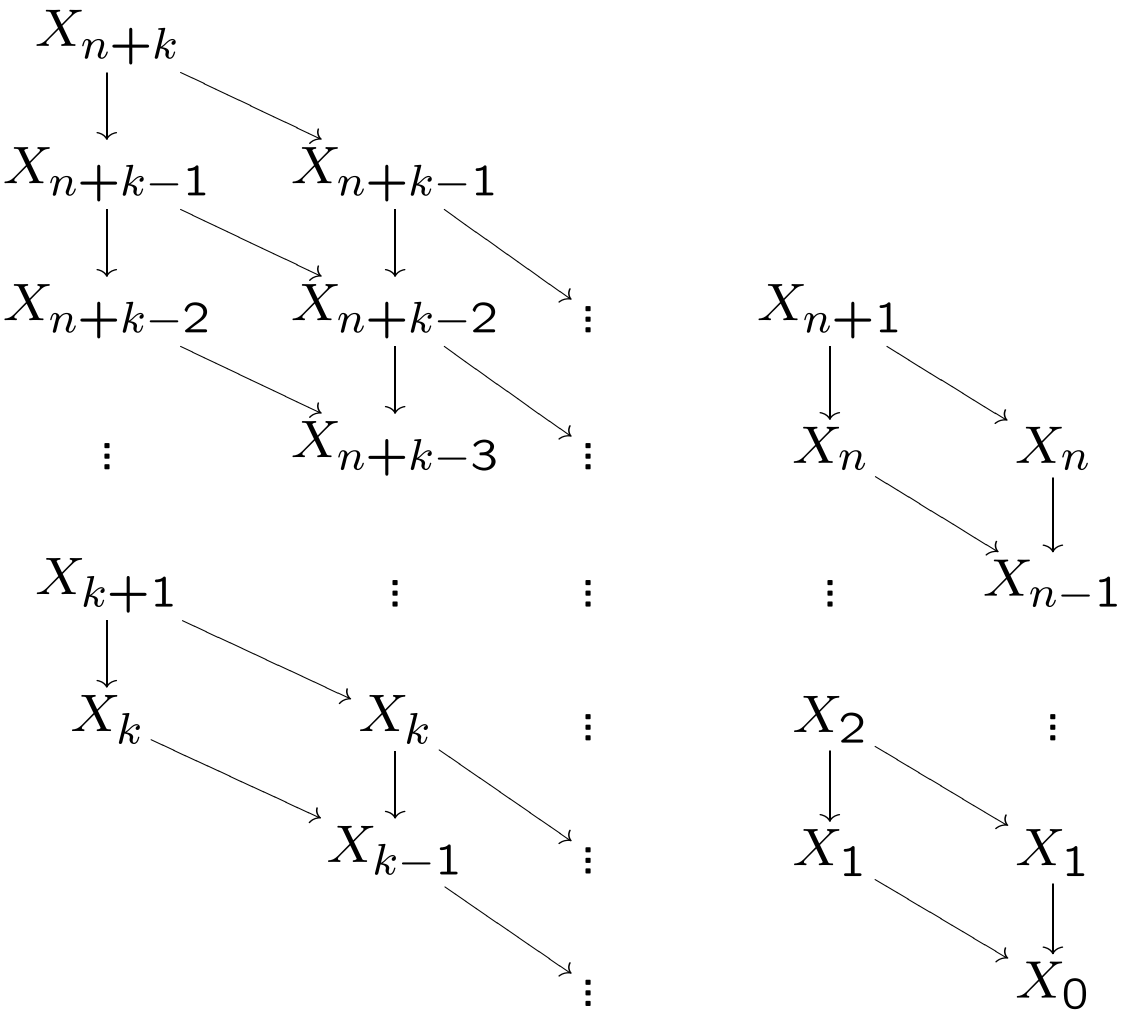
Example: Let $\mathcal{A} := \{K(\mathbb{F}_p, r)\}_{r=1}^{\infty}$ in $\mathcal{M} = \mathcal{T}op_*$ (or in spectra), and write $\Theta^{\mathbb{F}_p} := \Theta^{\mathcal{A}}$.
A dual $\mathbb{F}_p$-mapping algbera consists of products of $\mathbb{F}_p$-GEMs, acted on by maps between such.

n-**Truncated mapping algebras:**

Def: Apply n-th Postnikov section functor to $\mathfrak{X}$ yields an n-A-*mapping algebra* $P^n\mathfrak{X}$.

Example: When $A = S^1$, $P^0\mathfrak{M}_{\mathcal{A}}Y$ is just the Π-algebra $\pi_* Y$, equipped with an action of Whitehead products and compositions.
$P^1\Theta_A =$ track category of maps between spheres, and $P^1\mathfrak{M}_{\mathcal{A}}Y =$ *secondary* Π-*algebra* of Y.

Note: The n-S^1-mapping algebras describe n-*stems*:



Obstructions and higher homotopy operations:

For good A (such as $S^n \in \mathcal{T}op_*$):

<u>Theorem</u> (Baues-B.): There is a one-to-one correspondence between n-A-mapping algebras and $(n+1)$-truncated simplicial A-resolutions (in the sense of Dwyer-Kan-Stover), up to weak equivalence.

<u>Theorem</u> (B.): The obstruction to realizing a Π_A-algebra Λ by an $(n+1)$-truncated simplicial resolution $W_\bullet$ is an n-th order A-homotopy operation $\langle\langle\Lambda\rangle\rangle$.

<u>Theorem</u> (B.-Dwyer-Goerss): The n-th obstruction to realizing a Π-algebra Λ by an n-Postnikov simplicial resolution $W_\bullet$ is the $(n+1)$-st k-invariant $\beta_{n+1} \in H^{n+2}(\Lambda; \Omega^n\Lambda)$ for $\pi_*^A W_\bullet$.

<u>Theorem</u> (B.-Johnson-Turner): There is a natural homomorphism

$$\Phi_n : [\Sigma^{n-1}\mathbf{W}_{n+1}, \mathbf{W}_0] \to H^{n+2}(\Lambda; \Omega^n\Lambda)$$

taking the n-th order homotopy operation $\langle\langle\Lambda\rangle\rangle$ to the k-invariant β_{n+1}.

Universal n-th order homotopy operations:

There are similar theories for distinguishing between n-A-mapping algebras and $(n+1)$-truncated simplicial A-resolutions of a Π_A-algebra Λ, as well as interpreting the obstructions both cohomologically and as higher A-homotopy operations.

Summary of results so far:

(a) For small n, it is possible to re-interpret the $(\mathcal{S}, \mathcal{O})$-$k$-invariants of $\mathcal{M}$ as Baues-Wirsching-type cohomology classes, from which we can read off (long) Toda brackets.

It is not clear how one can recover other $(n+1)$-st order homotopy operations.

(b) If we look at the fragment of a homotopy theory $\mathcal{M}$ given by maps out of (or into) a fixed object A, its (co)products and suspensions (or loops), we can define an explicit "universal $(n+1)$-st order homotopy operation" to determine its n-th stem with respect to A.

Goal: To extend this to all of $\mathcal{M}$, at least for $\mathcal{M} = \mathcal{T}op_*$.