Composing Lawvere Theories

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Motivation and background

We describe a uniform framework for composing PROPs and (countable) Lawvere theories. There are several reasons for why we wish to do this.

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- Quantum Information In the Abramsky-Coecke approach to Quantum Information, one uses various interacting algebras/coalgebras and their string diagrammatic representation to model aspects of quantum mechanics. Expressing those algebraic structures as composites of simpler ones allows us to gain insight into the normal forms of their graphical representation, greatly simplifying reasoning using those structures and assisting with the creation of automated reasoning tools (e.g. Quantomatic: http://dream.inf.ed.ac.uk/projects/quantomatic).

Outline

1. Recall the definition of distributive laws between monads (in Cat)
2. Define Lawvere theories
3. Define PROPs
4. Describe Span(Mon) as a setting for composing strict monoidal categories
5. Describe Prof(Mon) as a setting for composing strict monoidal categories with extra structure
6. Show that Lawvere theories can be composed in Prof(Mon)
7. Discuss the generation of distributive laws and look at an example

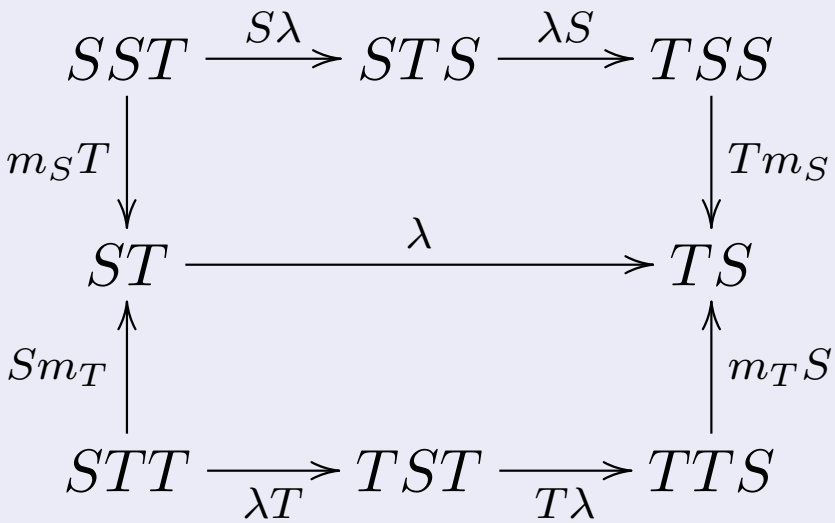
Distributive Laws

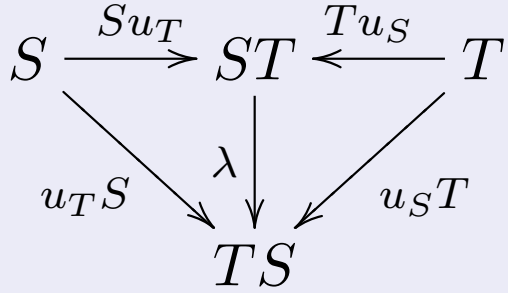
Distributive Law

Let $T, S : \mathcal{C} \to \mathcal{C}$ be monads. A *distributive law* of S over T is a nat. trans.

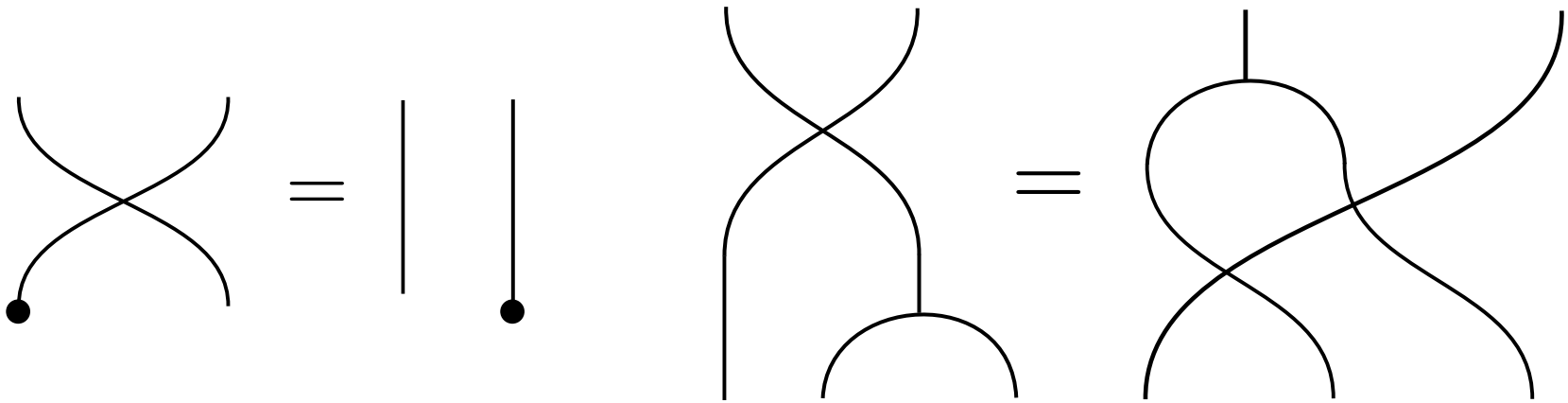
$$\lambda : ST \to TS$$

making the following diagrams commute:





The axioms, visually



Lawvere Theories and PROPs

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A famous example:

$$\mathrm{Mod}(\mathrm{FreeGrp}^{\mathrm{op}}, \mathrm{Set}) \simeq \mathrm{Grp}.$$

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For example, the category $\mathbb{F}$ is a PROP. Its algebras in $\mathcal{C}$ are the commutative monoids in $\mathcal{C}$.

Span and Prof

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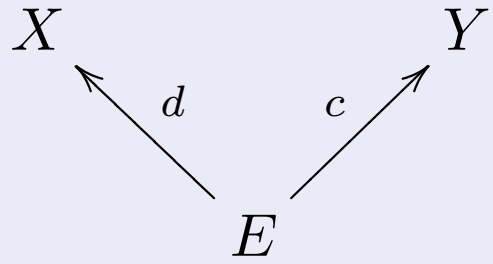
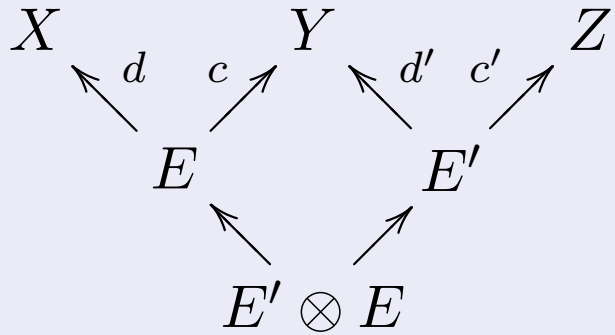


Composition of 1-cells is given by pullback:



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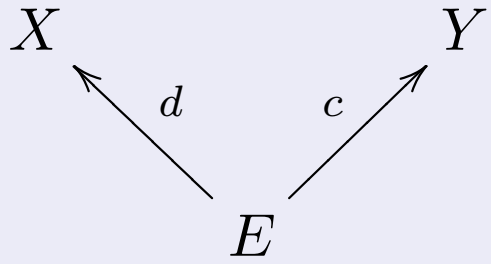

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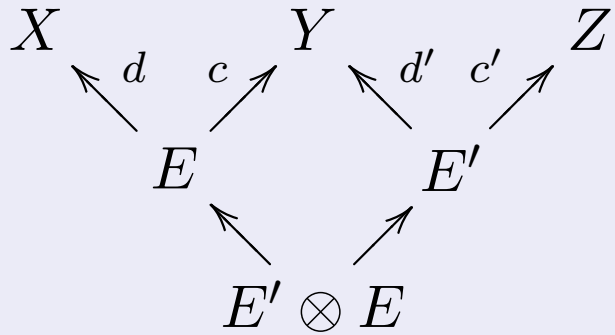
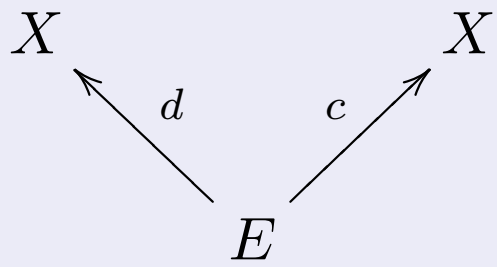


- 2-cells: morphisms of spans

Take $\mathcal{C} = \mathrm{Mon}$. The monads in $\mathrm{Span}(\mathrm{Mon})$ are strict monoidal categories:

A monad is a span

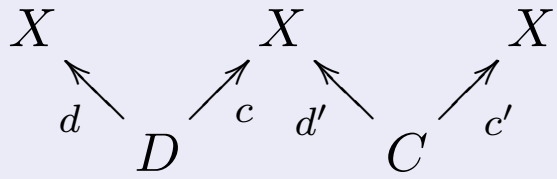


- X is the set of objects;
- E is the set of morphisms;
- d and c are the domain and codomain maps;
- $E \otimes E \to E$ is composition and $X \to E$ is given by the identities.
- monoidal structure comes from Mon.

Thus categories can be composed using distributive laws [Rosebrugh, Wood 2002]

Distributive laws in Span($\mathcal{C}$):

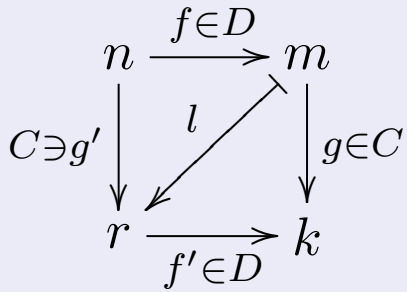
Let $\mathcal{C}$ and $\mathcal{D}$ be categories, with $\mathrm{ob}\mathcal{C} = X = \mathrm{ob}\mathcal{D}$ viewed as spans:



A *distributive law of* $\mathcal{C}$ *over* $\mathcal{D}$ is a $\mathcal{C}$-morphism

$$l : C \otimes D \to D \otimes C$$

which we depict by



This 2-cell interacts nicely with the multiplications and units of the monads C and D - i.e. with the composition and identities of the underlying categories:

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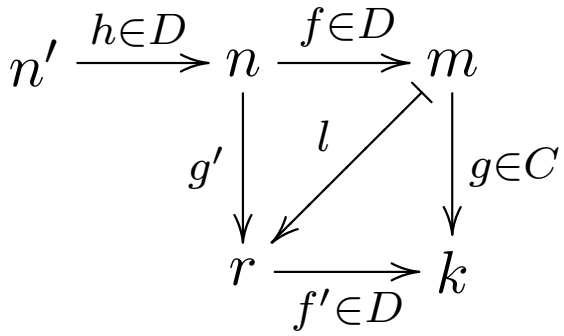

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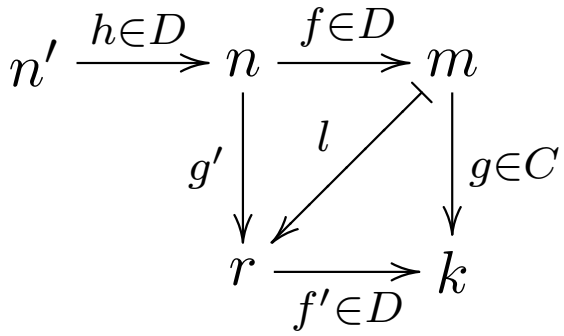

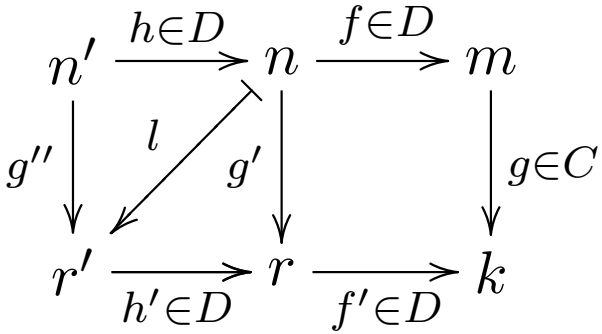

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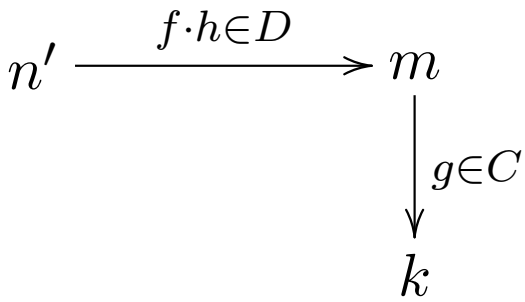

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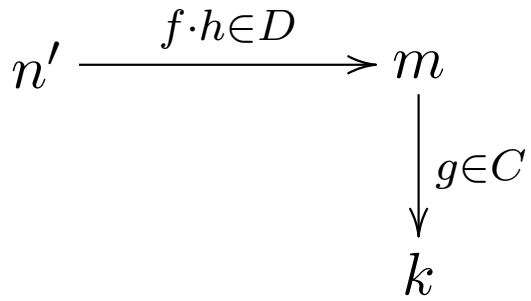

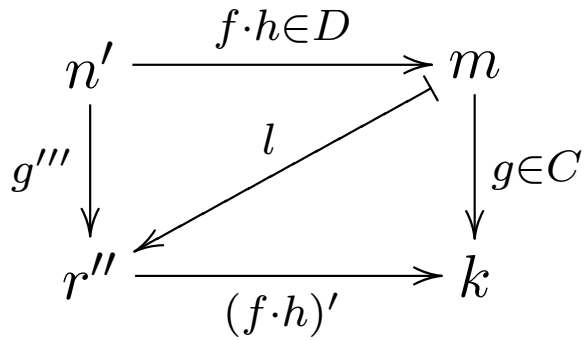

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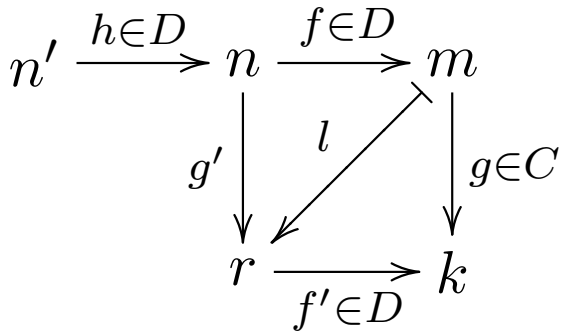

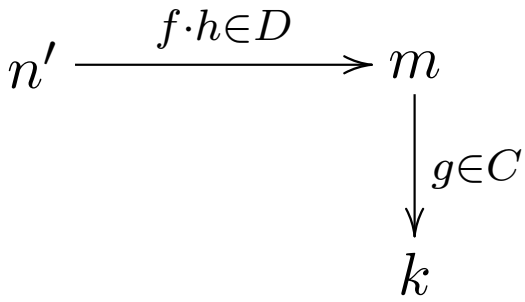

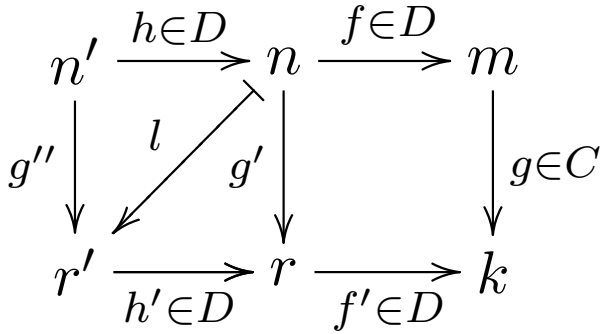

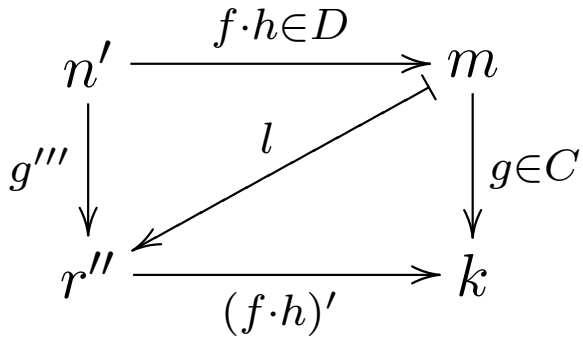

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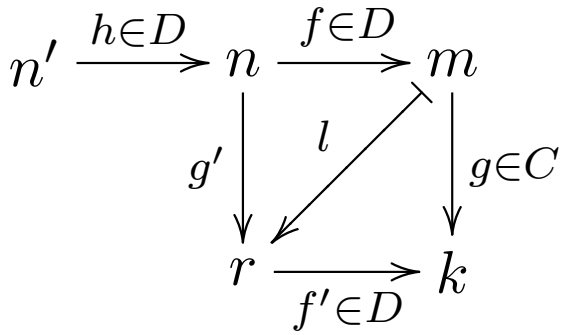

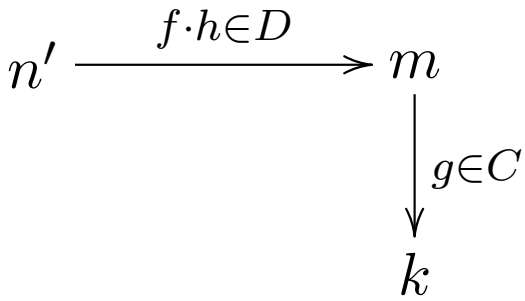

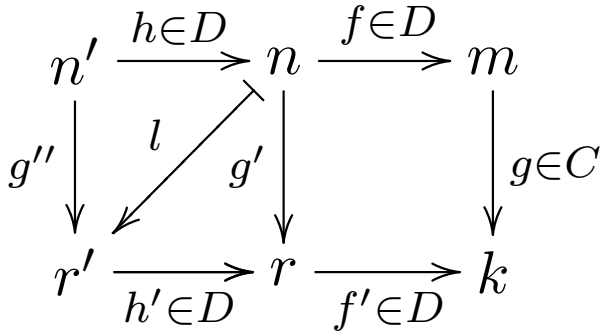

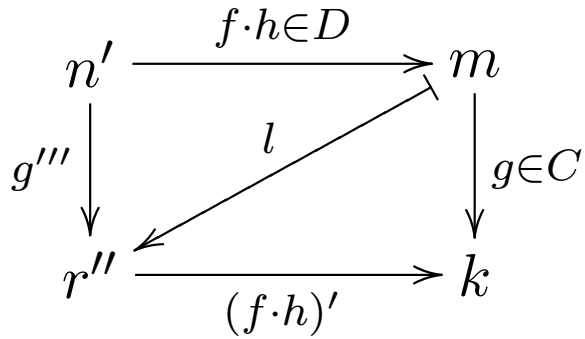

$$r' = r'',\ (f \cdot h)' = f' \cdot h',\ g''' = g''.$$

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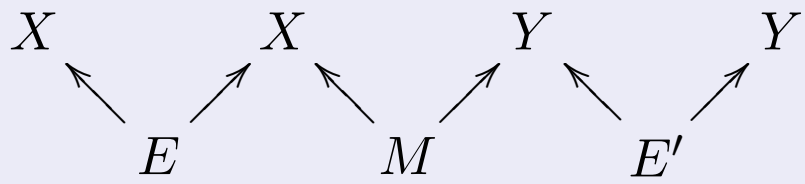

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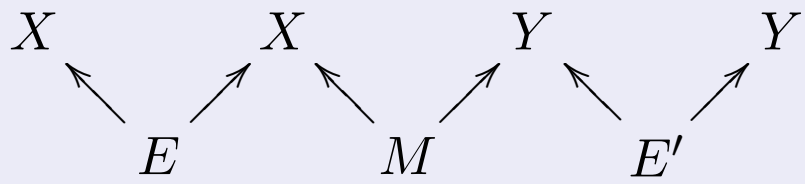

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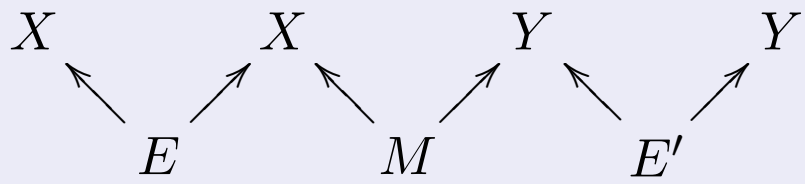

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- 2-cells: bimodule homomorphisms.

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- Multiplication and unit are given by the composition and identities of the underlying category;

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- With R and S rings, a 1-cell between them in $\mathrm{Prof}(\mathrm{Ab})$ is just an R-S-bimodule.
- Composition is the tensor product of bimodules.

Take $\mathcal{C} = \mathrm{Mon}$. And let L be a Lawvere theory. Then $L : \mathbb{F}^{\mathrm{op}} \longrightarrow \mathbb{F}^{\mathrm{op}}$ is a monad on $\mathbb{F}^{\mathrm{op}}$ in $\mathrm{Prof}(\mathrm{Mon})$:

- Multiplication and unit are given by the composition and identities of the underlying category;
- Actions: $L \otimes \mathbb{F}^{\mathrm{op}} \longrightarrow L, \quad (\pi, l) \mapsto l \cdot \pi.$

The following Theorem allows us to express various categories with structure as monads in Prof(Mon):

Theorem

Let $\mathcal{C}$ and $\mathcal{D}$ be *identity-on-objects* *categories. There is a 1-1 correspondence between* *functors*

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[S. Lack, 2004]

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Lack's Theorem about composite algebras for PROPs carries through with practically no changes to (countable) Lawvere theories.

Theorem

An $L \otimes_{\mathbb{C}^{\mathrm{op}}} L'$-algebra structures on A consists of an L-algebras structure on A, and L'-algebra structure on A such that for all $f : n \to m \in L$, $g : m \to r \in L'$

$$\begin{array}{ccc} A^n & \xrightarrow{f} & A^m \\ {\scriptstyle g'}\downarrow & & \downarrow{\scriptstyle g} \\ A^k & \xrightarrow{f'} & A^r \end{array}$$

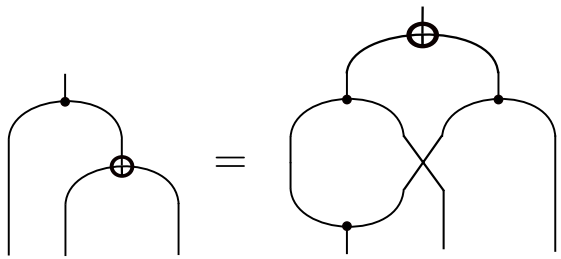
commutes.

Example

Let L be the Lawvere theory for semi-groups and L' the Lawvere theory for Abelian semigroups, given by the opposite of the category of free semigroups/abelian semigroups (on finitely many generators).
We can generate a distributive law

$$L \otimes_{\mathbb{F}^{\mathrm{op}}} L' \to L' \otimes_{\mathbb{F}^{op}} L$$

from the following equation



The composite theory is the Lawvere theory for semi-rings.

Summary and future work

- We saw that Prof(Mon) provides a good setting for composing categories with different types of monoidal structure.

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- To construct a collection of examples of composite PROPs/Lawvere theories we need the ability to generate distributive laws from (nice) rewriting systems.

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