Categorical methods for Hopf algebra theory

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[A1] A.L. Agore, "*Limits for coalgebras, bialgebras and Hopf algebras*", to appear in Proc. Amer. Math. Soc., arXiv:1003.0318v1

[A2] A.L. Agore, "*Categorical constructions for Hopf algebras*", to appear in Commun. in Algebra, arXiv:0905.2613v3

[A3] A.L. Agore, "*Monomorphisms of coalgebras*", Colloq. Math. 120 (2010), 149-155, arXiv:0908.2959v3

In what follows k is a field.

2 problems from Sweedler's book:

- The forgetful functor $U : k\text{-HopfAlg} \to k\text{-CoAlg}$ has a left adjoint

Takeuchi's theorem(1970): there exists a free Hopf algebra generated by a coalgebra, i.e. U has a left adjoint

- The forgetful functor $V : k\text{-HopfAlg} \to k\text{-Alg}$ has a right adjoint

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Proposition

The category k-HopfAlg has coequalizers: if $f, g : B \to A$ are two Hopf algebra maps and I is the two-sided ideal generated by $\{f(b) - g(b) \mid b \in B\}$ then $(A/I, \pi)$ is the coequalizer of the morphisms (f, g) in k-HopfAlg, where $\pi : A \to A/I$ is the canonical projection.

- The forgetful functor $V : k$-HopfAlg $\to k$-Alg (respectively $V' : k$-HopfAlg $\to k$-BiAlg) preserves coequalizers.

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- The forgetful functor $V : k$-HopfAlg $\to k$-Alg (respectively $V' : k$-HopfAlg $\to k$-BiAlg) preserves coequalizers.

Proposition

The categories k-Alg, k-BiAlg and k-HopfAlg have coproducts. Moreover, the forgetful functor $V : k\text{-HopfAlg} \to k\text{-Alg}$ (respectively $V' : k\text{-HopfAlg} \to k\text{-BiAlg}$) preserves coproducts.

Takeuchi also indicates the coproducts above (H.E. Porst - private communication) in a lemma, with no proof.

Proof.

(sketch) Let $(A_l)_{l\in I}$ be a family of Hopf algebras. We denote by $\left(\coprod_{l\in I} A_l, (j_l)_{l\in I}\right)$ the coproduct in k-Alg
$\coprod_{l\in I} A_l$ is actually a Hopf algebra with the antipode given by the unique algebra map such that the following diagram commutes:

$$\begin{array}{ccc} H_l & \xrightarrow{q_l} & H \\ {\scriptstyle S_l}\downarrow & & \downarrow{\scriptstyle S} \\ H_l^{opcop} & \xrightarrow{q_l} & H^{opcop} \end{array} \qquad (1)$$

□

Corollary

The category k-HopfAlg is cocomplete and the forgetful functor $V : k\text{-HopfAlg} \to k\text{-Alg}$ (respectively $V' : k\text{-HopfAlg} \to k\text{-BiAlg}$) preserves colimits.

The category k-HopfAlg:

- is cocomplete
- it has a generator (Pareigis, Sweedler)
- is co-locally small (H.E. Porst - locally presentable)

Since the forgetful functor $U : k\text{-HopfAlg} \to k\text{-Alg}$ preserves colimits we obtain, from The Special Adjoint Functor Theorem:

Theorem

([A2]) There exist a cofree Hopf algebra on every algebra (bialgebra), i.e. the forgetful functor $V : k\text{-HopfAlg} \to k\text{-Alg}$ (respectively $V' : k\text{-HopfAlg} \to k\text{-BiAlg}$) has a right adjoint.

Remark

Later on, A. Chirvasitu gave the construction of the cofree Hopf algebra on an algebra (A. Chirvasitu, J. Algebra **323** *(2010), 1593-1606).*

- **Equalizers in k-HopfAlg (Andruskiewitsch and Devoto)**

Proposition

The category k-HopfAlg has equalizers: if $f, g : C \to D$ are two Hopf algebra maps then (E, i) is the equalizer of the pair (f, g) in the category k-HopfAlg, where:

$$E = \{c \in C \mid c_{(1)} \otimes f(c_{(2)}) \otimes c_{(3)} = c_{(1)} \otimes g(c_{(2)}) \otimes c_{(3)}\}$$

and $i : E \to C$ is the canonical inclusion.

- **Products in k-HopfAlg**
  In [A1], the following is proved:

Theorem

The categories k-CoAlg, k-BiAlg and k-HopfAlg have products.

Related results Prof. H.E. Porst

Proof (sketch):

Let $\left(H_i, m_i, \eta_i, \Delta_i, \varepsilon_i, S_i\right)_{i\in I}$ be a family of Hopf algebras.

- Consider $(\prod_{i\in I} H_i, (\pi_i)_{i\in I})$ the product in ${}_k\mathcal{M}$ and $(K(\prod_{i\in I} H_i), p)$ the cofree coalgebra on $\prod_{i\in I} H_i$
- Let D be the sum of all subcoalgebras E of $K\left(\prod_{i\in I} H_i\right)$ such that $\pi_i \circ p \circ j_E$ is a coalgebra map for all $i \in I$, where $j_E : E \to K(\prod_{i\in I} H_i)$ is the canonical inclusion.

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- D is a Hopf algebra with $\eta : k \to D$, $m : D \otimes D \to D$ and $S : B^{op,cop} \to B$ such that the following diagrams commute for all $i \in I$, where $\varphi_i = \pi_i \circ p \circ j$:

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$$\begin{array}{ccc} k & & \\ {\scriptstyle \eta}\downarrow & \searrow^{\eta_i} & \\ D & \xrightarrow{\varphi_i} & H_i \end{array} \qquad \begin{array}{ccc} D \otimes D & & \\ {\scriptstyle m}\downarrow & \searrow^{m_i \circ (\varphi_i \otimes \varphi_i)} & \\ D & \xrightarrow{\varphi_i} & H_i \end{array} \tag{2}$$

$$\begin{array}{ccc} D & \xrightarrow{\varphi_i} & H_i \\ {\scriptstyle S}\uparrow & & \uparrow{\scriptstyle S_i} \\ D^{op,cop} & \xrightarrow{\varphi_i} & H_i^{op,cop} \end{array} \tag{3}$$

- **Limits in k-HopfAlg**
  Let J be a small category, $F : J \to k$-HopfAlg be a functor
  - We denote by $Hom(J)$ the set of all morphisms of J
  - Given a morphism $f \in J$ we denote by $dom(f)$ and $cod(f)$ the domain, respectively the codomain of f.

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- $(\Pi_{j \in J} F(j), (p_j)_{j \in J})$, $(\Pi_{u \in Hom(J)} F(cod(u)), (p_u)_{u \in Hom(J)})$ be the product in k-HopfAlg of the families $(F(j))_{j \in J}$, respectively $(F(cod(u)))_{u \in Hom(J)}$

- Let $f, g : \Pi_{j \in J} F(j) \to \Pi_{u \in Hom(J)} F(cod(u))$ be the unique Hopf algebra maps such that $p_u \circ f = p_{cod(u)}$ and $p_u \circ g = F(u) \circ p_{dom(u)}$ for all $u \in Hom(J)$

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Theorem

Let J be a small category, $F : J \to k$-HopfAlg be a functor. We define
$D = \{x \in \Pi_{j\in J} F(j) | x_{(1)} \otimes f(x_{(2)}) \otimes x_{(3)} = x_{(1)} \otimes g(x_{(2)}) \otimes_{(3)}\}$ *and* $e : D \to \Pi_{j\in J} F(j)$ *the canonical inclusion. Then, the pair* $\big(D, (\varphi_j = p_j \circ e)_{j\in J}\big)$ *is the limit of the functor F*

- **Applications**

Lemma

Monomorphisms and epimorphisms of Hopf algebras are not necessarily injective, respectively surjective maps (A. Chirvasitu, J. Algebra **323** *(2010), 1593-1606).*

- $S : H \to H^{op,cop}$ is both a monomorphism and an epimorphism of Hopf algebras

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From the existence of the two adjoint functors:

- Left adjoint functor for $U : k\text{-HopfAlg} \to k\text{-CoAlg}$
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Corollary

- *A morphism of Hopf algebras is a monomorphism iff it is a monomorphism when viewed as a map of coalgebras.*
- *A morphism of Hopf algebras is an epimorphism iff it is an epimorphism when viewed as a map of algebras.*

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Theorem

[A3] *Let* $f : H \to L$ *be a Hopf algebra map. The following are equivalent:*

1) $f : K \to L$ *is a Hopf algebra monomorphism;*
2) $H^0(N, H) = H^0(N, L)$*, for any* (H, H)*-bicomodule* N*;*
3) $\sum_{i \in I} \varepsilon(x^i) y^i = \sum_{i \in I} x^i \varepsilon(y^i)$*, for all* $\sum_{i \in I} x^i \otimes y^i \in H \Box_L H$*.*

where $H^0(N, H)$ is the first cohomology group of H with coefficients in N (Doi),
$H^0(N, H) = \{\gamma \in N^* | n_{<-1>} \gamma(n_{<0>}) = \gamma(n_{[0]}) n_{[1]}, \forall n \in N\}$

Theorem

Let $g : H \to L$ be a Hopf algebra map. The following are equivalent:

1) *$g : H \to L$ is a Hopf algebra epimorphism;*
2) *$l \otimes_R 1_S = 1_S \otimes_R s$, for any $s \in S$;*
3) *The map $\varepsilon_L : L \otimes_H L \to L$, $\varepsilon_L(l_1 \otimes_H l_2) = l_1 l_2$ is injective (hence an isomorphism in ${}_L\mathcal{M}_L$).*

THANK YOU