The Number of Countable Models in a Grothendieck Toposes

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In 1961 Vaught a question:

> *Can it be proved, without the use of the continuum hypothesis, that there exists a complete theory having exactly $\aleph_1$ non-isomorphic denumerable models?*

This is one of the oldest open problems in model theory and the statement that it has a negative answer has become known as **Vaught's Conjecture**. I.e. that

Conjecture (Vaught Conjecture)

Every first order theory T one of the following holds:

- *T has at most $\aleph_0$ many countable models.*
- *T has a perfect set of countable models.*

Morley's Theorem

One of the most important results in the study of Vaught's conjecture is:

Theorem (Morley)

For each countable language L and each sentence $T \in \mathcal{L}_{\omega_1,\omega}(L)$ one of the following holds:

- *T has at most $\aleph_1$ many countable models.*
- *T has a perfect set of countable models.*

This result expanded scope of Vaught's conjecture from first order logic to sentences of $\mathcal{L}_{\omega_1,\omega}$.

Goal

In this talk we discuss the following generalization of Morley's Theorem:

Theorem (A)

Assuming $\mathbf{\Pi}^1_3$-determinacy, whenever (C, J_C) is a countable site, L is a countable language and $T \in \mathcal{L}_{\omega_1,\omega}(L)$ is a sentence, one of the following holds:

- *T has at most $\aleph_1$ many countable models in the category of sheaves on (C, J_C).*
- *T has a perfect set of countable models in the category of sheaves on (C, J_C).*

Countable Sites

Definition

We say a site (C, J_C) is *countable* if

- C is a countable category (i.e. $|\text{morph}(C)| \leq \omega$).
- $J_C(A)$ is countable for each object A of C.

From now on

- (C, J_C) will be a countable site.
- $Sep(C, J_C)$ is the category of separated presheaves on (C, J_C).
- $Sh(C, J_C)$ is the category of sheaves on (C, J_C).

Definition of $\mathcal{L}_{\omega_1,\omega}(L)$

We let L be a countable multi-sorted language.

Definition

Recall that $\mathcal{L}_{\omega_1,\omega}(L)$ is the smallest collection of formulas such that

- $L \subseteq \mathcal{L}_{\omega_1,\omega}(L)$.
- $\mathcal{L}_{\omega_1,\omega}(L)$ is closed under negation ($\neg$).
- $\mathcal{L}_{\omega_1,\omega}(L)$ is closed under finite existential ($\exists x$) and universal ($\forall x$) quantification.
- $\mathcal{L}_{\omega_1,\omega}(L)$ is closed under infinite disjunctions ($\bigvee$) and conjunctions ($\bigwedge$) so long as each subformula contains only a finite number of free variables.

Models in a Category

Definition

A *model M of L in a category X* consists of the following:

- For each sort $S \in L$ there is an object $S^M \in \mathrm{obj}(X)$
- For each relation $R \in L$ of signature $(S_1, \ldots, S_n)$ there is a subobject $R^M \subseteq S_1^M \times \cdots \times S_n^M$
- For each function symbol $f \in L$ of signature $(S_1, \ldots, S_n) \to S$ there is a morphism $f^M : S_1^M \times \cdots \times S_n^M \to S^M$

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We will be most interested in models in the categories $Sep(C, J_C)$ and $Sh(C, J_C)$.

Lemma

The sheafification functor $\mathbf{a} : Sep(C, J_C) \to Sh(C, J_C)$ extends to a map from models in $Sep(C, J_C)$ to models in $Sh(C, J_C)$ (by applying $\mathbf{a}$ to each component of the model).

Countability of Models

Definition

A model M of the (countable) language L in $Sh(C, J_C)$ is *countable* if, for each sort S, S^M is a countable sheaf.

Of course for this to make sense we need to define what a countable sheaf is.

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Unfortunately, unlike in case of "countable sets", there are four distinct notions which have claim to the name of "countable sheaf".

Definition of Purely Countable Sheaves

Definition

A sheaf A over (C, J_C) is *purely countable* if $A(x)$ is countable for each $x \in \text{obj}(C)$.

In other words A is purely countable if it is isomorphic to a sheaf which, when expressed as a set, has countable transitive closure.

Unfortunately purely countable sheaves lack some properties we would hope a notion of countability would have. Such as:

Lemma

There is a site (C, J_C) such that the natural number object in $Sh(C, J_C)$ is not purely countable in any standard model of set theory.

Countably Generated Sheaves

Definition

A sheaf A is *countable generated* if there is an A^* such that

- A^* is a separated presheaf for (C, J_C).
- $A^*(x)$ is countable for each object x of C.
- The sheafification of A^* is isomorphic to A.

A countably generated sheaf is a direct analog of a separable metric space (i.e. a metric space with a countable dense subset).

Lemma

If C has only countably many objects, then natural number object of $Sh(C, J_C)$ is countably generated.

Lemma

For every sheaf A there is a forcing extension in which A is countably generated.

Monic Countable and Epi Countable

The next two notions of countable are are preserved under arbitrary equivalences of categories. Let $\mathbb{N} = \coprod_{i \in \omega} 1$ be a natural number object in $Sh(C, J_C)$.

Definition

A sheaf A is *monic countable* if there is a monomorphism

$$m : A \rightarrowtail \mathbb{N}$$

Definition

A sheaf A is *epi countable* if there is an epimorphism

$$e : \mathbb{N} \twoheadrightarrow A$$

Countability and Countably Generated

Note that being countably generated is the most general of these four notions and we have the following:

Lemma

If A is a purely countable sheaf then A is countably generated.

Lemma

If A is a monic countable or epi countable sheaf then A is countably generated.

Proof.

This follows from the fact that (C, J_C) is a countable site. □

Theorem

For any language L there is a countable language L^* and a sentence σ_L of $\mathcal{L}_{\omega_1,\omega}(L^*)$ such that:

- *The category of set models of σ_L is equivalent to the category of models of L in $Sep(C, J_C)$.*
- *A model M in $Sh(C, J_C)$ is countably generated if and only if there is a model M^* in $Sep(C, J_C)$ such that*
  - *The image of M^* under the equivalence in (1) is a countable model.*
  - *The sheafification of M^* is isomorphic to M.*

To simplify notation we won't distinguish between a model of L in $Sep(C, J_C)$ and the corresponding (set) model of σ_L.

Space of Countable Models

Definition

Let Str_{L^*} be the collection of (set) L^* models whose underlying set is ω along with the σ-algebra generated by sets of the form

$$\{M : M \models \varphi(n_1, \ldots, n_i),\ n_1, \ldots, n_i \in \omega,\ \varphi \in \mathcal{L}_{\omega_1,\omega}(L)\}$$

In particular Str_{L^*} is Borel isomorphic to 2^ω.

We let $Mod(\sigma_L) \subseteq Str_{L^*}$ be the (Borel) collection of those structures which satisfy σ_L.

Complexity of Countable Models

Now we can classify how "complicated" the space of countable models of a sentence of $\mathcal{L}_{\omega_1,\omega}(L)$ is.

Definition

Suppose $T \in \mathcal{L}_{\omega_1,\omega}(L)$ is a sentence. Define

- $Pure(T)$ to be the collection of models $M \in Mod(\sigma_L)$ such that the sheafification of M is *purely countable* and satisfies T.

We also define $Gen(T)$, $Monic(T)$, and $Epi(T)$ similarly except with *purely countable* replaced by *countably generated*, *monic countable* and *epi countable* respectively.

Theorem

For every sentence $T \in \mathcal{L}_{\omega_1,\omega}(L)$, $Pure(T)$, $Gen(T)$, $Monic(T)$ and $Epi(T)$ are $\mathbf{\Sigma}^1_2$ subsets of $Mod(\sigma_L)$

Complexity of Equivalence Relation

Definition

If $M, N \in Mod(\sigma_L)$ say $M \equiv_L N$ if the sheafification of M is isomorphic to the sheafification of N.

In particular, counting the number of "countable" models in $Sh(C, J_C)$ corresponds to counting the number of equivalence classes under $\equiv_L$.

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In particular, counting the number of "countable" models in $Sh(C, J_C)$ corresponds to counting the number of equivalence classes under $\equiv_L$.

Lemma

$\equiv_L$ is a $\mathbf{\Sigma}^1_2$-equivalence relation on $Mod(\sigma_L)$.

and if we restrict to purely countable models we get

Lemma

For every sentence $T \in \mathcal{L}_{\omega_1,\omega}(L)$, the relation $M \equiv_L N$ on $Pure(T)$ is a $\mathbf{\Sigma}^1_1$-relation.

Number of Purely Countable Models

Corollary

For every sentence $T \in \mathcal{L}_{\omega_1,\omega}(L)$ one of the following hold:

- *There are at most $\aleph_1$ many purely countable models of T in $Sh(C, J_C)$.*
- *There is a perfect set of purely countable models of T in $Sh(C, J_C)$.*

Proof.

Because under *ZF* every $\mathbf{\Sigma}^1_1$ equivalence relation (on a $\mathbf{\Sigma}^1_2$ set of reals) has either a perfect set of equivalence classes or at most $\aleph_1$ many equivalence classes. □

$\mathbf{\Sigma}^1_2$-Equivalence Relations

Unfortunately the axioms of *ZF* aren't, in general, sufficient to determine the number of equivalence classes of a $\mathbf{\Sigma}^1_2$ relation. But, if we assume $\mathbf{\Pi}^1_3$-determinacy then we have the following result:

Theorem (Harrington, Sami, Shelah)

Under $\mathbf{\Pi}^1_3$ determinacy, if a $\mathbf{\Sigma}^1_2$ equivalence relation on a $\mathbf{\Sigma}^1_2$ subset of $Mod(\sigma_L)$ does not have a perfect set of equivalence classes, it has at most $\aleph_1$ many equivalence classes.

Number of Countable Models

Corollary

Assuming $\mathbf{\Pi}^1_3$ *determinacy, for every sentence* $T \in \mathcal{L}_{\omega_1,\omega}(L)$ *one of the following hold:*

- *There are at most* $\aleph_1$ *many countably generated models of* T *in* $Sh(C, J_C)$*.*
- *There is a perfect set of countably generated models of* T *in* $Sh(C, J_C)$*.*

The same holds for monic countable and epi countable models of T*.*

Generalized Vaught's Conjecture

Definition

A site (C, J_C) has the *Vaught Property* if for each of the four notions of countable and every sentence $T \in \mathcal{L}_{\omega_1,\omega}(L)$ one of the following two holds:

- There is a perfect set of countable models of T in $Sh(C, J_C)$.
- There are at most $\aleph_0$ many countable models of T in $Sh(C, J_C)$.

Conjecture (Sheaf Vaught's Conjecture)

Every countable site (C, J_C) has the Vaught property.

Example of Vaught Property

Example

Consider the site (C_ω, J_{C_ω}) where

- C_ω is the set $\{0, 1, 2, \dots\} = \omega$ considered as a category.
- $J_{C_\omega}(n)$ is trivial for all $n \in \omega$.

Then for every sentence $T \in \mathcal{L}_{\omega_1,\omega}(L)$ the one of the following holds

- T has at most one countable (set) model and hence at most one countable model in $Sh(C_\omega, J_{C_\omega})$ (for any of the notions of countable).
- T has at least two countable (set) models and hence a perfect set of countable models in $Sh(C_\omega, J_{C_\omega})$ (for any of the notions of countable).

Hence (C_ω, J_{C_ω}) has the Vaught property.

Thank You